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Differential geometry on the space of connections via graphs and projective limits

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Abstract

In a quantum mechanical treatment of gauge theories (including general relativity), one is led to consider a certain completion $\overline{\mathcal{A}/\mathcal{G}}$ of the space $\mathcal{A}/\mathcal{G}$ of gauge equivalent connections. This space serves as the quantum configuration space, or, as the space of all Euclidean histories over which one must integrate in the quantum theory. $\overline{\mathcal{A}/\mathcal{G}}$ is a very large space and serves as a "universal home" for measures in theories in which the Wilson loop observables are well defined. In this paper, $\overline{\mathcal{A}/\mathcal{G}}$ is considered as the projective limit of a projective family of compact Hausdorff manifolds, labelled by graphs (which can be regarded as "floating lattices" in the physics terminology). Using this characterization, differential geometry is developed through algebraic methods. In particular, we are able to introduce the following notions on $\overline{\mathcal{A}/\mathcal{G}}$: differential forms, exterior derivatives, volume forms, vector fields and Lie brackets between them, divergence of a vector field with respect to a volume form, Laplacians and associated heat kernels and heat kernel measures. Thus, although $\overline{\mathcal{A}/\mathcal{G}}$ is very large, it is small enough to be mathematically interesting and physically useful. A key feature of this approach is that it does not require a background metric. The geometrical framework is therefore well suited for diffeomorphism invariant theories such as quantum general relativity.

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1. Introduction

Theories of connections are playing an increasingly important role in the current description of all fundamental interactions of Nature (including gravity [1]). They are also of interest from a purely mathematical viewpoint. In particular, many of the recent advances in the understanding of the topology of low-dimensional manifolds have come from these theories.

In the standard functional analytic approach, developed in the context of the Yang–Mills theory, one equips the space of connections with the structure of a Hilbert–Riemann manifold (see, e.g., [12]). This structure is gauge-invariant. However, the construction uses a fixed Riemannian metric on the underlying space–time manifold. For diffeomorphism invariant theories – such as general relativity – this is, unfortunately, a serious drawback. A second limitation of this approach comes from the fact that, so far, it has led to relatively few examples of interesting, gauge-invariant measures on spaces of connections, and none that is diffeomorphism invariant. Hence, to deal with theories such as quantum general relativity, a gauge and diffeomorphism invariant extension of these standard techniques is needed.

For the functional integration part of the theory, such an extension was carried out in a series of papers over the past two years [4,5,13,14,21,6]. (For earlier work with the same philosophy, see [16].) The purpose of this article is to develop differential geometry along the same lines. Our constructions will be generally motivated by certain heuristic results in a nonperturbative approach quantum gravity based on connections, loops and holonomies [2,22]. Reciprocally, our results will be useful in making this approach rigorous [9,3] in that they provide the well-defined measures and differential operators that are needed in a rigorous treatment. There is thus a synergetic exchange of ideas and techniques between the heuristic and rigorous treatments.

As background material, we will first present some physical considerations and then discuss our approach from a mathematical perspective.

Fix an n-dimensional manifold M and consider the space $\mathcal{A}$ of smooth connections on a given principal bundle $B(M, G)$ over M. Following the standard terminology, we will refer to G as the structure group and denote the space of smooth vertical automorphisms of $B(M, G)$ by $\mathcal{G}$. This $\mathcal{G}$ is the group of *local* gauge transformations. If M is taken to be a Cauchy surface in a Lorentzian space–time, the quotient $\mathcal{A}/\mathcal{G}$ serves as the physical configuration space of the classical gauge theory. If M represents the Euclidean space–time, $\mathcal{A}/\mathcal{G}$ is the space of physically distinct classical histories. Because of the presence of an infinite number of degrees of freedom, to go over to quantum field theory, one has to enlarge $\mathcal{A}/\mathcal{G}$ appropriately. Unfortunately, since $\mathcal{A}/\mathcal{G}$ is nonlinear, with complicated topology, a canonical mathematical extension is not available. For example, the simple idea of substituting the smooth connections and gauge transformations in $\mathcal{A}/\mathcal{G}$ by distributional ones does not work because the space of distributional connections does not support the action of distributional local gauge transformations.

Recently, one such extension was introduced [4] using the basic representation theory of $\mathbb{C}^\star$-algebras. The ideas underlying this approach can be summarized as follows. One first considers the space $\mathcal{HA}$ of functions on $\mathcal{A}/\mathcal{G}$ obtained by taking finite complex linear combinations of finite products of Wilson loop functions $W_\alpha(A)$ around closed loops α. (Recall that the Wilson loop functions are traces of holonomies of connections around closed loops; $W_\alpha(A) = \mathrm{Tr}\,\mathcal{P}\ \oint_\alpha A\,\mathrm{d}l$. Since they are gauge-invariant, they project down unambiguously to $\mathcal{A}/\mathcal{G}$.) $\mathcal{HA}$ can then be completed in a natural fashion to obtain a $\mathbb{C}^\star$-algebra $\overline{\mathcal{HA}}$. This is the algebra of configuration observables. Hence, to obtain the Hilbert space of physical states, one has to select an appropriate representation of $\overline{\mathcal{HA}}$. It turns out that every cyclic representation of $\overline{\mathcal{HA}}$ by operators on a Hilbert space is of a specific type [4]: The Hilbert space is simply $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu)$ for some regular, Borel measure μ on a certain completion $\overline{\mathcal{A}/\mathcal{G}}$ of $\mathcal{A}/\mathcal{G}$ and, as one might expect of configuration operators, the Wilson loop operators act just by multiplication. Therefore, the space $\overline{\mathcal{A}/\mathcal{G}}$ is a candidate for the required extension of the classical configuration. To define physically interesting operators, one needs to develop differential geometry on $\overline{\mathcal{A}/\mathcal{G}}$. For example, the momentum operators would correspond to suitable vector fields on $\overline{\mathcal{A}/\mathcal{G}}$ and the kinetic term in the Hamiltonian would be given by a Laplacian. The problem of introducing these operators is coupled to that of finding suitable measures on $\overline{\mathcal{A}/\mathcal{G}}$ because these operators have to be essentially self-adjoint on the underlying Hilbert space.

From a mathematical perspective, $\overline{\mathcal{A}/\mathcal{G}}$ is just the Gel'fand spectrum of the Abelian $\mathbb{C}^\star$-algebra $\overline{\mathcal{HA}}$; it is a compact, Hausdorff topological space and, as the notation suggests, $\mathcal{A}/\mathcal{G}$ is densely embedded in it. The basic techniques for exploring the structure of this space were introduced in [5]. It was shown that $\overline{\mathcal{A}/\mathcal{G}}$ is very large: in particular, every connection on *every* G-bundle over M defines a point in $\overline{\mathcal{A}/\mathcal{G}}$. (Note incidentally that this implies that $\overline{\mathcal{A}/\mathcal{G}}$ is independent of the initial choice of the principal bundle $B(M, G)$ made in the construction of the holonomy algebra $\mathcal{HA}$.) Furthermore, there are points which do not correspond to *any* smooth connection; these are the generalized connections (defined on generalized principal G-bundles [18]) which are relevant only to the quantum theory. Finally, there is a precise sense in which this space provides a "universal home" for measures that arise from lattice gauge theories [11]. In specific theories, such as Yang–Mills, the support of the relevant measures is likely to be significantly smaller. For diffeomorphism invariant theories, on the other hand, there are indications that it would be essential to use the whole space. In particular, it is known that $\overline{\mathcal{A}/\mathcal{G}}$ admits large families of measures which are invariant under the induced action of Diff(M) [5,13,14,9,6] and therefore likely to feature prominently in nonperturbative quantum general relativity [9,3]. Many of these are *faithful* indicating that all of $\overline{\mathcal{A}/\mathcal{G}}$ would be relevant to quantum gravity.

Thus, the space $\overline{\mathcal{A}/\mathcal{G}}$ is large enough to be useful in a variety of contexts. Indeed, at first sight, one might be concerned that it is too large to be physically useful. For example, by construction, it has the structure *only* of a topological space; it is not even a manifold. How can one then hope to introduce the basic quantum operators on $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu)$? In absence of a well-defined manifold structure on the quantum configuration space, it may seem impossible to introduce vector fields on it, let alone the Laplacian or the operators needed in quantum gravity! Is there a danger that $\overline{\mathcal{A}/\mathcal{G}}$ is so large that it is mathematically uninteresting?

Fortunately, it turns out that, although it is large, $\overline{\mathcal{A}/\mathcal{G}}$ is "controllable". The key reason is that the $\mathbb{C}^\star$-algebra $\overline{\mathcal{HA}}$ is rather special, being generated by the Wilson loop observables. As a consequence, its spectrum $\overline{\mathcal{A}/\mathcal{G}}$ can also be obtained as the projective limit of a projective family of compact, Hausdorff, *analytic manifolds* [5,13,14,21,6]. Standard projective constructions therefore enable us to induce on $\overline{\mathcal{A}/\mathcal{G}}$ various notions from differential geometry. Thus, it appears that a desired balance is struck: While it is large enough to serve as a "universal home" for measures, $\overline{\mathcal{A}/\mathcal{G}}$ is, at the same time, small enough to be mathematically interesting and physically useful. This is the main message of this paper.

The material is organized as follows. In Section 2, we recall from [21,6] the essential results from projective techniques. In Section 3, we use these results to construct three projective families of compact, Hausdorff, analytic manifolds, and show that $\overline{\mathcal{A}/\mathcal{G}}$ can be obtained as the projective limit of one of these families. Since the members of the family are all manifolds, each is equipped with the standard differential geometric structure. Using projective techniques, Sections 4 and 5 then carry this structure to the projective limits. Thus, the notions of forms, volume forms, vector fields and their Lie-derivatives and divergence of vector fields with respect to volume forms can be defined on $\overline{\mathcal{A}/\mathcal{G}}$. The vector fields which are compatible with the measure (in the sense that their divergence with respect to the measure is well defined) lead to essentially self-adjoint momentum operators in the quantum theory. In Section 6, we turn to Riemannian geometry. Given an additional structure on the underlying manifold M – called an edge-metric – we define a Laplacian operator on the $\mathbb{C}^2$-functions on $\overline{\mathcal{A}/\mathcal{G}}$ and construct the associate heat kernels as well as the heat kernel measures. In Section 7, we point out that $\overline{\mathcal{A}/\mathcal{G}}$ admits a natural (degenerate) contravariant metric and use it to introduce a Laplace-like operator. Since this construction does not use any background structure on M, the action of the operator respects diffeomorphism invariance. It could thus define a natural observable in diffeomorphism invariant quantum theories. Another example is a third-order differential operator representing the "volume observable" in quantum gravity. Section 8 puts the analysis of this paper in the context of the earlier work in the subject.

A striking aspect of this approach to geometry on $\overline{\mathcal{A}/\mathcal{G}}$ is that its general spirit is the same as that of noncommutative geometry and quantum groups: Although, there is no underlying differentiable manifold, geometrical notions can be developed by exploiting the properties of the *algebra* of functions. On the one hand, the situation with respect to $\overline{\mathcal{A}/\mathcal{G}}$ is simpler because the algebra in question is Abelian. On the other hand, we are dealing with very large, infinite-dimensional spaces. As indicated above, a primary motivation for this work comes from the mathematical problems encountered in a nonperturbative approach to quantum gravity [2] and our results can be used to solve a number of these problems [9,3]. However, there are some indications that, to deal satisfactorily with the issue of "framed loops and graphs" that may arise in regularization of certain operators, one may have to replace the structure group $SU(2)$ with its quantum version $SU(2)_q$. Our algebraic approach is well suited for an eventual extension along these lines.

2. Projective techniques: General framework

In this section, we recall from [21,6] some general results on projective limits which will be used in the rest of the paper.

We begin with the notion of a projective family. The first object we need is a set L of *labels*. The only structure L has is the following: it is a partially ordered, directed set. That is, it is a set equipped with a relation "$\geq$" such that, for all γ, γ' and γ'' in L we have:

$$\gamma \geq \gamma\,; \quad \gamma \geq \gamma' \text{ and } \gamma' \geq \gamma \Rightarrow \gamma = \gamma'\,; \quad \gamma \geq \gamma' \text{ and } \gamma' \geq \gamma'' \Rightarrow \gamma \geq \gamma''\,; \tag{1}$$

and, given any $\gamma', \gamma'' \in L$, there exists $\gamma \in L$ such that

$$\gamma \geq \gamma' \quad \text{and} \quad \gamma \geq \gamma''\,. \tag{2}$$

A *projective family* $(\mathcal{X}_\gamma, p_{\gamma\gamma'})_{\gamma,\gamma' \in L}$ consists of sets $\mathcal{X}_\gamma$ indexed by elements of L, together with a family of surjective *projections*,

$$p_{\gamma\gamma'} : \mathcal{X}_{\gamma'} \to \mathcal{X}_\gamma, \tag{3}$$

assigned uniquely to pairs (γ', γ) whenever $\gamma' \geq \gamma$ such that

$$p_{\gamma\gamma'} \circ p_{\gamma'\gamma''} = p_{\gamma\gamma''}. \tag{4}$$

A familiar example of a projective family is the following. Fix a locally convex, topological vector space V. Let the label set L consist of finite-dimensional subspaces γ of $V^\star$, the topological dual of V. This is obviously a partially ordered and directed set. Every γ defines a unique subspace $\tilde{\gamma}$ of V via: $\tilde{v} \in \tilde{\gamma}$ iff $\langle v, \tilde{v}\rangle = 0\, \forall v \in \gamma$. The projective family can now be constructed by setting $\mathcal{X}_\gamma = V/\tilde{\gamma}$. Each $\mathcal{X}_\gamma$ is a finite-dimensional vector space and, for $\gamma' \geq \gamma$, $p_{\gamma\gamma'}$ are the obvious projections. Integration theory over infinite-dimensional topological spaces can be developed starting from this projective family [17,15]. In this paper, we wish to consider projective families which are in a certain sense complementary to this example and which are tailored to the kinematically nonlinear spaces of interest.

In our case, $\mathcal{X}_\gamma$ will all be topological, compact, Hausdorff spaces and the projections $p_{\gamma\gamma'}$ will be continuous. The resulting pairs $(\mathcal{X}_\gamma, p_{\gamma\gamma'})_{\gamma,\gamma' \in L}$ are said to constitute a *compact Hausdorff projective family*. In the application of this framework to gauge theories, the labels γ can be thought of as "floating" lattices (i.e., which are not necessarily rectangular) and the members $\mathcal{X}_\gamma$ of the projective family, as the spaces of configurations/histories associated with these lattices. The continuum theory will be recovered in the (projective) limit as one considers lattices with increasing number of loops of arbitrary complexity.

Note that in the projective family there will, in general, be no set $\overline{\mathcal{X}}$ which can be regarded as the largest, from which we can project to any of the $\mathcal{X}_\gamma$. However, such a set does emerge in an appropriate limit, which we now define. The *projective limit* $\overline{\mathcal{X}}$ of a projective family $(\mathcal{X}_\gamma, p_{\gamma\gamma'})_{\gamma,\gamma' \in L}$ is the subset of the Cartesian product $\times_{\gamma \in L}\mathcal{X}_\gamma$ that satisfies certain consistency conditions:

$$\overline{\mathcal{X}} := \{(x_\gamma)_{\gamma \in L} \in \times_{\gamma \in L}\mathcal{X}_\gamma : \gamma' \geq \gamma \Rightarrow p_{\gamma\gamma'}x_{\gamma'} = x_\gamma\}. \tag{5}$$

(In applications to gauge theory, this is the limit that gives us the continuum theory.) One can show that $\overline{\mathcal{X}}$, endowed with the topology that descends from the Cartesian product, *is itself a compact, Hausdorff space*. Finally, as expected, one can project from the limit to any member of the family: we have

$$p_\gamma \;: \overline{\mathcal{X}} \to \mathcal{X}_\gamma, \quad p_\gamma((x_{\gamma'})_{\gamma' \in L}) := x_\gamma \,. \tag{6}$$

Next, we introduce certain function spaces. For each γ consider the space $\mathbb{C}^0(\mathcal{X}_\gamma)$ of the complex valued, continuous functions on $\mathcal{X}_\gamma$. In the union

$$\bigcup_{\gamma \in L} \mathbb{C}^0(\mathcal{X}_\gamma)$$

let us define the following equivalence relation. Given $f_{\gamma_i} \in \mathbb{C}^0(\mathcal{X}_{\gamma_i})$, $i = 1, 2$, we will say

$$f_{\gamma_1} \sim f_{\gamma_2} \quad \text{if } p^\star_{\gamma_1\gamma_3} f_{\gamma_1} = p^\star_{\gamma_2\gamma_3} f_{\gamma_2} \tag{7}$$

for every $\gamma_3 \geq \gamma_1, \gamma_2$, where $p^\star_{\gamma_1\gamma_3}$ denotes the pull-back map from the space of functions on $\mathcal{X}_{\gamma_1}$ to the space of functions on $\mathcal{X}_{\gamma_3}$. (Note that to be equivalent, it is in fact sufficient that equality (7) holds *just for one* $\gamma_3 \geq \gamma_1, \gamma_2$.)

Using the equivalence relation we can introduce the set of *cylindrical functions* associated with the projective family $(\mathcal{X}_\gamma, p_{\gamma\gamma'})_{\gamma,\gamma' \in L}$,

$$\mathrm{Cyl}^0(\overline{\mathcal{X}}) := \Big(\bigcup_{\gamma \in L} \mathbb{C}^0(\mathcal{X}_\gamma) \Big) / \sim \,. \tag{8}$$

The quotient just gets rid of a redundancy: pull-backs of functions from a smaller set to a larger set are now identified with the functions on the smaller set. Note that an element of $\mathrm{Cyl}^0(\overline{\mathcal{X}})$ determines, through the projections (6), a function on $\overline{\mathcal{X}}$. Hence, there is a natural embedding

$$\mathrm{Cyl}^0(\overline{\mathcal{X}}) \;\to\; \mathbb{C}^0(\overline{\mathcal{X}}),$$

which is dense in the sup-norm. Thus, modulo the completion, $\mathrm{Cyl}^0(\overline{\mathcal{X}})$ may be identified with the algebra of continuous functions on $\overline{\mathcal{X}}$ [6]. This fact will motivate, in Section 3, our definition of $\mathbb{C}^n$ functions on the projective completion.

Next, let us illustrate how one can introduce interesting structures on the projective limit. Since each $\mathcal{X}_\gamma$ in our family as well as the projective limit $\overline{\mathcal{X}}$ is a compact, Hausdorff space, we can use the standard machinery of measure theory on each of these spaces. The natural question is: What is the relation between measures on $\mathcal{X}_\gamma$ and those on $\overline{\mathcal{X}}$? To analyze this issue, let us begin with a definition. Let us assign to each $\gamma \in L$ a regular Borel, probability (i.e., normalized) measure, μ_γ on $\mathcal{X}_\gamma$. We will say that this constitutes a *consistent family of measures* if

$$(p_{\gamma\gamma'})_\star \, \mu_{\gamma'} = \mu_\gamma \,. \tag{9}$$

Using this notion, we can now characterize measures on $\overline{\mathcal{X}}$ [6].

Theorem 1. *Let $(\mathcal{X}_\gamma, p_{\gamma\gamma'})_{\gamma\gamma' \in L}$ be a compact, Hausdorff projective family and $\overline{\mathcal{X}}$ be its projective limit:*

(a) *Suppose μ is a regular, Borel, probability measure on $\overline{\mathcal{X}}$. Then μ defines a consistent family of regular, Borel, probability measures, given by*

$$\mu_\gamma := p_{\gamma_\star}\mu. \tag{10}$$

(b) *Suppose $(\mu_\gamma)_{\gamma,\gamma' \in L}$ is a consistent family of regular, Borel, probability measures. Then there is a unique regular, Borel, probability measure μ on $\overline{\mathcal{X}}$ such that $(p_\gamma)_\star\, \mu = \mu_\gamma$.*

(c) *μ is faithful if $\mu_\gamma := (p_\gamma)_\star\, \mu$ is faithful for every $\gamma \in L$.*

This is an illustration of the general strategy we will follow in Sections 4–7 to introduce interesting structures on the projective limit; they will correspond to families of *consistent* structures on the projective family.

3. Projective families for spaces of connections

We will now apply the general techniques of Section 2 to obtain three projective families, each member of which is a compact, Hausdorff, analytic manifold.

Fix an n-dimensional, analytic manifold M and a smooth principal fiber bundle $B(M, G)$ with the structure group G which we will assume to be a compact and connected Lie group. Let $\mathcal{A}$ denote the space of smooth connections on B and $\mathcal{G}$ the group of smooth vertical automorphisms of B (i.e., the group of local gauge transformations). The projective limits of the three families will provide us with completions $\overline{\mathcal{A}}$, $\overline{\mathcal{G}}$ and $\overline{\mathcal{A}/\mathcal{G}}$ of the spaces $\mathcal{A}$, $\mathcal{G}$ and $\mathcal{A}/\mathcal{G}$. As the notation suggests, $\overline{\mathcal{A}/\mathcal{G}}$ will turn out to be naturally isomorphic with the Gel'fand spectrum of the holonomy algebra of [4], mentioned in Section 1.

The label set of all three families will be the same. Section 3.1 introduces this set and Sections 3.2–3.4 discuss the three families and their projective limits. The results of this section follow from a rather straightforward combination of the results of [5,13,14,21,6]. Therefore, we will generally skip the detailed proofs and aim at presenting only the final structure which is used heavily in the subsequent sections.

3.1. Set of labels

The set L of labels will consist of graphs in M. To obtain a precise characterization of this set, let us begin with some definitions.

By an unparametrized oriented analytic edge in M we shall mean an equivalence class of maps

$$e : [0, 1] \to M, \tag{11}$$

where two maps e and e' are considered as equivalent if they differ only by a reparametrization, or, more precisely if e' can be written as

$$e' = e \circ f, \quad \text{where } f : [0, 1] \to [0, 1], \tag{12}$$

is an analytic orientation preserving bijection. We will also consider unoriented edges for which the requirement that f-preserve orientation will be dropped. The end points of an edge will be referred to as *vertices*. (If the edges are oriented, each e has a well-defined initial and a well-defined final vertex.) A (oriented) *graph* γ in M is a set of finite, unparametrized (oriented) analytic edges which have the following properties:

(1) every $e \in \gamma$ is diffeomorphic with the closed interval $[0, 1]$;
(2) if $e_1, e_2 \in \gamma$, with $e_1 \neq e_2$, the intersection $e_1 \cap e_2$ is contained in the set of vertices of e_1, e_2;
(3) every $e \in \gamma$ is at both sides connected with another element of γ.

(Note that the last condition ensures that each graph is closed.) The set of all the graphs in M will be denoted by L. This is our set of labels.

As we saw in Section 2, the set of labels must be a partially ordered, directed set. On our set of graphs, the partial order $\geq$ is defined just by the inclusion relation

$$\gamma' \geq \gamma \tag{13}$$

whenever each edge of γ can be expressed as a composition of edges of γ' *and* each vertex in γ is a vertex of γ'.

To see that the set is directed, we use the analyticity of edges: it is easy to check that, given any two graphs $\gamma_1, \gamma_2 \in L$, there exists $\gamma \in L$ such that

$$\gamma \geq \gamma_1 \quad \text{and} \quad \gamma \geq \gamma_2. \tag{14}$$

(In fact, given γ_1 and γ_2, there exists a *minimal* upper bound γ.) This property is no longer satisfied if one weakens the definition and only requires that the edges be smooth.

3.2. *The projective family for $\mathcal{A}$*

We are now ready to introduce our first projective family.

Fix a graph $\gamma \in L$. To construct the corresponding space $\mathcal{A}_\gamma$ in the projective family, restrict the bundle B to the bundle over γ, which we will denote by B_γ. Clearly, B_γ is the union of smooth bundles B_e over the edges of γ, $B_\gamma = \bigcup_{e\in\gamma} B_e$. For every edge $e \in \gamma$, any connection $A \in \mathcal{A}$ restricts to a smooth connection A_e on B_e. The collection $(A_e)_{e\in\gamma} =: A_{|\gamma}$ will be referred to as the restriction of A to γ. Denote by $\widehat{\mathcal{G}}^\gamma$ the subgroup of $\mathcal{G}$ which consists of those vertical automorphisms of B which act as the identity in the fibers of B over the vertices of γ. Now, since the action of $\mathcal{G}$ on $\mathcal{A}$ is equivariant with the restriction map $\mathcal{A} \to \mathcal{A}_{|\gamma}$, we can define the required space $\mathcal{A}_\gamma$ as

$$\mathcal{A}_\gamma := (\mathcal{A}/\widehat{\mathcal{G}}^\gamma)_\gamma. \tag{15}$$

Note that $\mathcal{A}_\gamma$ naturally decomposes into the Cartesian product

$$\mathcal{A}_\gamma = \times_{e\in\gamma}\mathcal{A}_e, \tag{16}$$

where $\mathcal{A}_e$ is defined by replacing γ in (15) with a single edge e. Next, let us equip $\mathcal{A}_\gamma$ with the structure of a differential manifold. Note first that, given an orientation of e, a component A_e of $(A_e)_{e\in\gamma} = A_\gamma \in \mathcal{A}_\gamma$, may be identified with the parallel transport map along the edge e which carries the fiber over its initial vertex into the fiber over its final vertex. Hence, if we fix over each vertex of γ a point in B and orient each edge of γ, we have natural maps

$$\Lambda_e : \mathcal{A}_e \to G \quad \text{and} \quad \Lambda_\gamma : \mathcal{A}_\gamma \to G^E, \tag{17}$$

where E is the number of edges in γ. The map can easily be shown to be a bijection. We shall refer to Λ_e (or Λ_γ) as a group valued chart for $\mathcal{A}_e$ or $(\mathcal{A}_\gamma)$. Now, since G is a compact, connected Lie group, G^E is a compact, Hausdorff, analytic manifold. Hence, the map Λ_γ can be used to endow $\mathcal{A}_\gamma$ with the same structure.

Finally, we introduce the required projection maps. Note first that, for each $\gamma \in L$, there is a natural projection map π_γ

$$\pi_\gamma : \mathcal{A} \to \mathcal{A}_\gamma \tag{18}$$

defined by (15), which is surjective. We now use this map to define projections $p_{\gamma\gamma'}$ between the members of our projective family. Let $\gamma' \geq \gamma$. Then, we set

$$p_{\gamma\gamma'} : \mathcal{A}_{\gamma'} \to \mathcal{A}_\gamma \tag{19}$$

to be the map defined by

$$\pi_\gamma = p_{\gamma\gamma'} \circ \pi_{\gamma'} . \tag{20}$$

We now have the following proposition.

Proposition 1.

(i) *The map Λ_γ of (17) is bijective and the analytic manifold structure defined on $\mathcal{A}_\gamma$ by Λ_γ does not depend on the initial choice of points in the fibers of B over the vertices of γ and orientation of the edges made in its definition;*

(ii) *for every pair of graphs $\gamma, \gamma' \in L$ such that $\gamma' \geq \gamma$, the map $p_{\gamma\gamma'}$ defined by* (18) *and* (20) *is surjective;*

(iii) *for any three graphs $\gamma, \gamma', \gamma'' \in L$ such that $\gamma'' \geq \gamma' \geq \gamma$,*

$$p_{\gamma\gamma''} = p_{\gamma\gamma'} \circ p_{\gamma'\gamma''} ; \tag{21}$$

and

(iv) *the maps $p_{\gamma\gamma'}$ are analytic.*

The proofs are straightforward. It is worth noting however, that to show the surjectivity in (i), one needs the assumption that the structure group G is connected [5]. On the other hand, compactness of G is not used directly in Proposition 1. Compactness is used, of course, in concluding that $\mathcal{A}_\gamma$ is compact.

Thus, we have introduced a projective family $(\mathcal{A}_\gamma, p_{\gamma\gamma'})_{\gamma,\gamma'\in L}$ of compact, Hausdorff, analytic manifolds, labelled by graphs in M. We will denote its projective limit by $\overline{\mathcal{A}}$.

We will conclude this subsection by presenting a characterization of $\overline{\mathcal{A}}$. Note first that a connection $A \in \mathcal{A}$ naturally defines a point $A_\gamma \in \mathcal{A}_\gamma$ for each $\gamma \in L$ and that the resulting family $(A_\gamma)_{\gamma\in L}$ represents a point in $\overline{\mathcal{A}}$. Hence, we have a natural map

$$\mathcal{A} \rightarrow \overline{\mathcal{A}}, \tag{22}$$

which is obviously an injection. There are however elements of $\overline{\mathcal{A}}$ which are not in the image of this map. In fact, "most of" $\overline{\mathcal{A}}$ lies outside $\mathcal{A}$. To represent a general element of $\overline{\mathcal{A}}$, we proceed as follows. Consider first a map I which assigns to each oriented edge e in M, an isomorphism

$$I(e) : B_{e_-} \rightarrow B_{e_+}, \tag{23}$$

between the fibers $B_{e_\pm}$ over $e_\pm$, the final and the initial end points of e. Suppose, that this map I satisfies the following two properties:

$$I(e^{-1}) = [I(e)]^{-1} \quad \text{and} \quad I(e_2 \circ e_1) = I(e_2) \circ I(e_1)\,, \tag{24}$$

whenever the composed path $e_2 \circ e_1$ is again analytic. (Here e^{-1} is the edge obtained from e by inverting its orientation and if $e_{1+} = e_{2-}$, $e_2 \circ e_1$ is the edge obtained by gluing edges e_2, e_1.) Then, we call I a *generalized parallel transport* in B. Let us denote the space of all these generalized parallel transports by $\mathcal{P}(B)$. Every element of the projective limit $\overline{\mathcal{A}}$ defines uniquely an element $I_{\bar{A}}$ of $\mathcal{P}(B)$. Indeed, let $\bar{A} = (A_\gamma)_{\gamma\in L} \in \overline{\mathcal{A}}$. For an oriented edge e in M pick any graph γ which contains e as the product of its edges (for some orientation) and define

$$I_{\bar{A}}(e) := H(A_\gamma, e)\,, \tag{25}$$

where the right-hand side stands for the (ordinary) parallel transport defined by $A_\gamma \in \mathcal{A}_\gamma$. From the definition of the projective limit $\overline{\mathcal{A}}$ it is easy to see that (25) gives rise to a well-defined map

$$\overline{\mathcal{A}} \ni \bar{A} \mapsto I_{\bar{A}} \in \mathcal{P}(B)\,. \tag{26}$$

Furthermore, it is straightforward to show the following properties of this map.

Proposition 2. *The map* (26) *defines a one-to-one correspondence between the projective limit* $\overline{\mathcal{A}}$ *and the space* $\mathcal{P}(B)$ *of generalized parallel transports in* B.

This characterization leads us to regard $\overline{\mathcal{A}}$, heuristically, as the configuration space of all possible "floating " lattices in M, prior to the removal of gauge freedom at the vertices (see (15)).

3.3. The projective family for $\mathcal{G}$

As we just noted, in the projective family constructed in the last section, there is still a remaining gauge freedom: given a graph γ, the restrictions of the vertical automorphisms of the bundle B to the vertices of γ still act nontrivially on $\mathcal{A}_\gamma$. In this subsection, we will construct a projective family $(\mathcal{G}_\gamma, p_{\gamma\gamma'})$ from these restricted gauge transformations. In the next section, we will use the two families to construct the physically relevant quotient projective family.

Given a graph γ, the restricted gauge freedom is the image of the following projection:

$$\tilde{\pi}_\gamma : \mathcal{G} \rightarrow \mathcal{G}/\widehat{\mathcal{G}}^\gamma =: \mathcal{G}_\gamma \, . \tag{27}$$

Clearly, the group $\mathcal{G}_\gamma$ has a natural action on $\mathcal{A}_\gamma$. Since $\mathcal{G}_\gamma$ consists essentially of the gauge transformations "acting at the vertices" of γ (up to the natural isomorphism) one can write $\mathcal{G}_\gamma$ as the cartesian product group

$$\mathcal{G}_\gamma = \times_{v\in \mathrm{Ver}(\gamma)}\mathcal{G}_v \, , \tag{28}$$

where $\mathcal{G}_v$ is, as before, the group of automorphisms of the fiber $\pi^{-1}(v) \subset B$ and $\mathrm{Ver}(\gamma)$ stands for the set of the vertices of γ. Now, each group $\mathcal{G}_v$ is isomorphic with the structure group G. Hence, if we fix a point in the fiber over each vertex of γ, we obtain an isomorphism

$$\tilde{\Lambda}_\gamma : \mathcal{G}_\gamma \rightarrow G^V \tag{29}$$

where V is the number of edges of γ. Finally, given any two graphs $\gamma' \geq \gamma$, the map $\tilde{\pi}_\gamma$ of (27) factors into

$$\tilde{\pi}_\gamma = p_{\gamma\gamma'} \circ \tilde{\pi}_{\gamma'}, \quad p_{\gamma\gamma'} : \mathcal{G}_{\gamma'} \rightarrow \mathcal{G}_\gamma \tag{30}$$

and hence defines the maps $p_{\gamma\gamma'}$ uniquely. It is easy to verify that this machinery is sufficient to endow $(\mathcal{G}_\gamma, p_{\gamma\gamma'})_{\gamma,\gamma'\in L}$ the structure of a compact, connected Lie group projective family. We have the following proposition.

Proposition 3.

(i) *The family $(\mathcal{G}_\gamma, p_{\gamma\gamma'})_{\gamma,\gamma'\in L}$ defined by* (27) *and* (30) *is a smooth projective family;*

(ii) *the maps $p_{\gamma\gamma'}$ are Lie group homomorphisms;*

(iii) *the projective limit $\overline{\mathcal{G}}$ of the family is a compact topological group with respect to the pointwise multiplication: let $(g_\gamma)_{\gamma\in L}, (h_\delta)_{\delta\in L} \in \overline{\mathcal{G}}$, then*

$$(g_\gamma)_{\gamma\in L}(h_\delta)_{\delta\in L} := (g_\gamma h_\gamma)_{\gamma\in L}; \tag{31}$$

(iv) *there is a natural topological group isomorphism*

$$\overline{\mathcal{G}} \rightarrow \times_{x\in M}\mathcal{G}_x, \tag{32}$$

where the group on the right-hand side is equipped with the product topology.

In view of the item (iv), we again have the expected embedding

$$\mathcal{G} \rightarrow \overline{\mathcal{G}}, \tag{33}$$

where the group $\mathcal{G}$ of automorphisms of B is identified with the subgroup consisting of those families $(g_x)_{x\in M} \in \overline{\mathcal{G}}$ which are smooth in x.

Let us equip $\mathcal{G}_\gamma = G^V$ with the measure $\mu_\gamma = (\mu_0)^V$, where μ_0 is the Haar measure on $\mathcal{G}$. Then it is straightforward to verify that $(\mu_\gamma)_{\gamma\in L}$ is a consistent family of measures in the sense of Section 2. Hence, it defines a regular, Borel Probability measure $\bar{\mu}_0$ on $\overline{\mathcal{G}}$. This is just the Haar measure on $\overline{\mathcal{G}}$. Thus, by enlarging the group $\mathcal{G}$ to $\overline{\mathcal{G}}$, one can obtain a compact group of generalized gauge transformations whose total volume is *finite* (actually, unit). This observation was first made by Baez [14].

3.4. The quotient $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ and the projective family for $\overline{\mathcal{A}/\mathcal{G}}$

In the last two subsections, we constructed two projective families. Their projective limits, $\overline{\mathcal{A}}$ and $\overline{\mathcal{G}}$ are the completions of the spaces $\mathcal{A}$ and $\mathcal{G}$ of smooth connections and gauge transformations. The action of $\mathcal{G}$ on $\mathcal{A}$ can be naturally extended to an action of $\overline{\mathcal{G}}$ on $\overline{\mathcal{A}}$. Indeed, let $(g_\gamma)_{\gamma\in L} \in \overline{\mathcal{G}}$ and $(A_\delta)_{\delta\in L} \in \overline{\mathcal{A}}$. Then, we set

$$(A_\delta)_{\delta\in L}(g_\gamma)_{\gamma\in L} := (A_\delta g_\delta)_{\delta\in L} \in \overline{\mathcal{A}}\,, \tag{34}$$

where $(A_\delta, g_\delta) \mapsto A_\delta g_\delta$ denotes the action of $\mathcal{G}_\delta$ in $\mathcal{A}_\delta$. Now, this action of $\overline{\mathcal{G}}$ on $\overline{\mathcal{A}}$ is continuous and $\overline{\mathcal{G}}$ is a compact topological group. Hence, the quotient $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ is a Hausdorff and compact space. This concludes the first part of this subsection.

In the second part, we will examine the spaces $\mathcal{A}_\gamma/\mathcal{G}_\gamma$. Note first that the projections $p_{\gamma\gamma'}$ defined in (20), (18) descend to the projections of the quotients:

$$p_{\gamma\gamma'} : \mathcal{A}_{\gamma'}/\mathcal{G}_{\gamma'} \to \mathcal{A}_\gamma/\mathcal{G}_\gamma\,. \tag{35}$$

We thus have a new compact, Hausdorff, projective family $(\mathcal{A}_\gamma/\mathcal{G}_\gamma, p_{\gamma\gamma'})_{\gamma,\gamma'\in L}$. This family can also be obtained directly from the quotient $\mathcal{A}/\mathcal{G}$ by a procedure which is analogous to the one used in Section 4.1: the space $\mathcal{A}_\gamma/\mathcal{G}_\gamma$ assigned to a graph γ is just the image of the restriction map

$$\pi_\gamma : \mathcal{A}/\mathcal{G} \to (\mathcal{A}/\mathcal{G})_{|\gamma} = \mathcal{A}_\gamma/\mathcal{G}_\gamma. \tag{36}$$

Therefore, it is natural to denote the projective limit of $(\mathcal{A}_\gamma/\mathcal{G}_\gamma, p_{\gamma\gamma'})_{\gamma,\gamma'\in L}$ is by $\overline{\mathcal{A}/\mathcal{G}}$.

The natural question now is: What is the relation between $\overline{\mathcal{A}/\mathcal{G}}$ and $\overline{\mathcal{A}}/\overline{\mathcal{G}}$? Note first that there is a natural map from $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ to $\overline{\mathcal{A}/\mathcal{G}}$, namely

$$\overline{\mathcal{A}}/\overline{\mathcal{G}} \ni [(A_\gamma)_{\gamma\in L}] \mapsto ([A_\gamma])_{\gamma\in L} \in \overline{\mathcal{A}/\mathcal{G}}, \tag{37}$$

where the square bracket denotes the operation of taking the orbit with respect to the corresponding group. Using the results of [21,6], it is straightforward to show the following proposition.

Proposition 4. *The map* (37) *defines a homeomorphism with respect to the quotient geometry on $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ and the projective limit geometry on $\overline{\mathcal{A}/\mathcal{G}}$.*

Finally, by combining the results of [13] and [6], one can show that the space $\overline{\mathcal{A}/\mathcal{G}}$ is naturally isomorphic to the Gel'fand spectrum of the holonomy $\mathbb{C}^\star$-algebra $\overline{\mathcal{HA}}$ introduced in Section 1. (Thus, there is no ambiguity in notation.) The space $\overline{\mathcal{A}/\mathcal{G}}$ thus serves as the quantum configuration space of the continuum gauge theory. This is the space of direct physical interest.

We conclude with a number of remarks.

(1) Since $\overline{\mathcal{A}/\mathcal{G}}$ is the Gel'fand spectrum of $\overline{\mathcal{HA}}$, it follows [5] that there is a natural embedding

$$\bigcup_{B'} (\mathcal{A}/\mathcal{G})_{B'} \rightarrow \overline{\mathcal{A}/\mathcal{G}}, \tag{38}$$

where $(\mathcal{A}/\mathcal{G})_{B'}$ denotes the quotient space of connections on a bundle B' and B' runs through all the G-principal fiber bundles over M. Thus, although it is not obvious from our construction, $\overline{\mathcal{A}/\mathcal{G}}$ is independent of the choice of the bundle B we made in the beginning; it is tied only to the underlying manifold M. (See [4,5,13,21] for the bundle independent definitions.)

(2) Each member $\mathcal{A}_\gamma$ and $\mathcal{G}_\gamma$ of the first two projective families, we considered is a compact, analytic manifold. Unfortunately, the same is not true of the quotients $\mathcal{A}_\gamma/\mathcal{G}_\gamma$ which constitute the third family since the quotient construction introduces kinks and boundaries. Because of this, while discussing differential geometry, we will regard $\overline{\mathcal{A}/\mathcal{G}}$ as $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ and deal with $\overline{\mathcal{G}}$-invariant structures on $\overline{\mathcal{A}}$. That is, it would be more convenient to work "upstairs" on $\overline{\mathcal{A}}$ even though $\overline{\mathcal{A}/\mathcal{G}}$ is the space of direct physical interest. This point was first emphasized by Baez [13,14].

(3) In the literature, one often fixes a base point x_0 in M and uses the subgroup $\mathcal{G}_{x_0}$ of $\mathcal{G}$ consisting of vertical automorphisms which act as the identity on the fiber over $x_0 \in M$ as the gauge group. In the present framework, this corresponds to considering the subgroups $\mathcal{G}_{\gamma,x_0} \subset \mathcal{G}_\gamma$ where γ run through L_{x_0}, the space of graphs which have x_0 as a vertex. $\overline{\mathcal{A}/\mathcal{G}}$ can be recovered by taking the quotient of the projective limit of this family by the natural action of the gauge group at the base point.

4. Elements of differential geometry on $\overline{\mathcal{A}}$

We are now ready to discuss differential geometry. We saw in Section 2 that one can introduce a measure on the projective limit by specifying a consistent family of measures on the members of the projective family. The idea now is to use this strategy to introduce on the projective limits various structures from differential geometry. The object of our primary interest is $\overline{\mathcal{A}/\mathcal{G}}$. However, as indicated above, we will first introduce geometric structures of $\overline{\mathcal{A}}$. Those structures which are invariant under the action of $\overline{\mathcal{G}}$ on $\overline{\mathcal{A}}$ will descend to $\overline{\mathcal{A}/\mathcal{G}} = \overline{\mathcal{A}}/\overline{\mathcal{G}}$ and provide us, in Section 5, with differential geometry on the quotient.

In Section 4.1, we introduce $\mathbb{C}^n$ differential forms on $\overline{\mathcal{A}}$ and, in Section 4.2, the $\mathbb{C}^n$ volume forms. Section 4.3 is devoted to $\mathbb{C}^n$ vector fields and their properties. Finally, in Section 4.4, we combine these results to show how vector fields can be used to define "momentum operators" in the quantum theory. While we will focus on the projective family introduced in Section 3.2, our analysis will go through for *any* projective family, the members of which are smooth compact manifolds.

Throughout this section $\mathbb{C}^n$ could in particular stand for $\mathbb{C}^\infty$ or $\mathbb{C}^\omega$.

4.1. Differential forms

Let us begin with functions.

Results of Section 2 imply that the projective limit $\overline{\mathcal{A}}$ of the family $(\mathcal{A}_\gamma, p_{\gamma\gamma'})_{\gamma,\gamma'\in L}$ is a compact Hausdorff space. Hence, we have a well-defined algebra $\mathbb{C}^0(\overline{\mathcal{A}})$ of continuous functions on $\overline{\mathcal{A}}$. We now want to introduce the notion of $\mathbb{C}^n$ functions on $\overline{\mathcal{A}}$. The problem is that $\overline{\mathcal{A}}$ does not have a natural manifold structure. Recall however that the algebra $\mathbb{C}^0(\overline{\mathcal{A}})$ could also be constructed directly from the projective family, without passing to the limit: We saw in Section 2 that $\mathbb{C}^0(\overline{\mathcal{A}})$ is naturally isomorphic with the algebra $\mathrm{Cyl}^0(\overline{\mathcal{A}})$ of cylindrical continuous functions. The idea now is to simply define differentiable functions on $\overline{\mathcal{A}}$ as cylindrical, differential functions on the projective family.

This is possible because each member $\mathcal{A}_\gamma$ of the family has the structure of an analytic manifold, and the projections $p_{\gamma\gamma'}$ are all analytic. Thus, we can define $\mathbb{C}^n$ cylindrical functions $\mathrm{Cyl}^n(\overline{\mathcal{A}})$ to be

$$\mathrm{Cyl}^n(\overline{\mathcal{A}}) \;:=\; \bigcup_{\gamma\in\Gamma} \mathbb{C}^n(\mathcal{A}_\gamma)/\sim, \tag{39}$$

where the equivalence relation is the same as in (7) of Section 2; as before, it removes the redundancy by identifying, if $\gamma' \geq \gamma$, the function f on $\mathcal{A}_\gamma$ with its pull-back f' on $\mathcal{A}_{\gamma'}$. Elements of $\mathrm{Cyl}^n(\overline{\mathcal{A}})$ will serve as the $\mathbb{C}^n$ functions on $\overline{\mathcal{A}}$. Note that if a cylindrical function $f \in \mathrm{Cyl}(\overline{\mathcal{A}})$ can be represented by a function $f_\gamma \in \mathbb{C}^n(\mathcal{A}_\gamma)$, then all the representatives of f are of the $\mathbb{C}^n$ differentiability class.

Next we consider higher-order forms. The idea is again to use an equivalence relation $\sim$ to "glue" differential forms on $(\mathcal{A}_\gamma)_{\gamma\in L}$ and obtain strings that can serve as differential forms on $\overline{\mathcal{A}}$. Consider $\bigcup_{\gamma\in\Gamma}\Omega(\mathcal{A}_\gamma)$, where $\Omega(\mathcal{A}_\gamma)$ denotes the Grassman algebra of all $\mathbb{C}^n$ sections of the bundle of differential forms on $\mathcal{A}_\gamma$. Let us introduce the equivalence relation $\sim$ by extending (7) in an obvious way:

$$\Omega(\mathcal{A}_{\gamma_1}) \ni \omega_{\gamma_1} \;\sim\; \omega_{\gamma_2} \in \Omega(\mathcal{A}_{\gamma_2}) \quad \text{iff} \quad p^\star_{\gamma_1\gamma'}\omega_{\gamma_1} \;=\; p^\star_{\gamma_2\gamma'}\omega_{\gamma_2} \tag{40}$$

for any $\gamma' \geq \gamma_1, \gamma_2$. (Again, if the equality above is true for a particular γ' then it is true for every $\gamma' \geq \gamma_1, \gamma_2$.) The set of differential forms on $\overline{\mathcal{A}}$ we are seeking is now given by

$$\Omega(\overline{\mathcal{A}}) \;:=\; \big(\bigcup_{\gamma\in\Gamma} \Omega(\mathcal{A}_\gamma)\big)/\sim. \tag{41}$$

Clearly, $\Omega(\overline{\mathcal{A}})$ contains well-defined subspaces $\Omega^m(\overline{\mathcal{A}})$ of m-forms. Since the pull-backs $p^\star_{\gamma\gamma'}$ commute with the exterior derivatives, there is a natural, well-defined exterior derivative operation d on $\overline{\mathcal{A}}$:

$$d : \Omega^n(\overline{\mathcal{A}}) \;\to\; \Omega^{n+1}(\overline{\mathcal{A}}). \tag{42}$$

One can use it to define and study the corresponding cohomology groups $H^n(\overline{\mathcal{A}})$.

Thus, although $\overline{\mathcal{A}}$ does not have a natural manifold structure, using the projective family and an algebraic approach to geometry, we can introduce on it $\mathbb{C}^n$ differential forms and exterior calculus.

4.2. *Volume forms*

Volume forms require a special treatment because they are not encompassed in the discussion of the previous section. To see this, recall that an element of $\Omega(\overline{\mathcal{A}})$ is an assignment of a consistent family of m-forms, for some *fixed* m, to each $\mathcal{A}_\gamma$, with $\gamma \geq \gamma_0$ for some γ_0. On the other hand, since a volume form on $\overline{\mathcal{A}}$ is to enable us to integrate elements of $\mathrm{Cyl}^0(\overline{\mathcal{A}})$, it should correspond to a consistent family of d_γ-forms on $(\mathcal{A}_\gamma)_{\gamma\in L}$, where d_γ is the dimension of the manifold $\mathcal{A}_\gamma$. That is, the rank of the form is no longer fixed but changes with the dimension of $\mathcal{A}_\gamma$. Thus, volume forms are analogous to the measures discussed in Section 2 rather than to the n-forms discussed above.

The procedure to introduce them is pretty obvious from our discussion of measures. A $\mathbb{C}^n$ *volume form* on $\overline{\mathcal{A}}$ will be a family $(\nu_\gamma)_{\gamma\in L}$, where each ν_γ is a $\mathbb{C}^n$ volume form with strictly positive volume on $\mathcal{A}_\gamma$, such that

$$\mathbb{C}^0(\mathcal{A}_{\gamma'}) \ni f_{\gamma'} \sim f_\gamma \in \mathbb{C}^0(\mathcal{A}_\gamma) \;\Rightarrow\; \int\limits_{\mathcal{A}_\gamma} f_\gamma \nu_\gamma = \int\limits_{\mathcal{A}_{\gamma'}} f_{\gamma'}\nu_{\gamma'} \tag{43}$$

for all $\gamma' \geq \gamma$ and all functions f_γ on $\mathcal{A}_\gamma$, where $f_{\gamma'} = p^\star_{\gamma\gamma'} f_\gamma$. Now, since $\mathcal{A}_\gamma$ are all compact, it follows from the discussion of measures in Section 2 that this volume form automatically defines a regular Borel measure, say ν, on $\overline{\mathcal{A}}$ and that this measure satisfies:

$$\int\limits_{\overline{\mathcal{A}}} [f_\gamma]_\sim \,\mathrm{d}\nu := \int\limits_{\mathcal{A}_\gamma} f_\gamma \nu_\gamma. \tag{44}$$

The most natural volume form μ_0 on $\overline{\mathcal{A}}$ is provided by the normalized, left and right invariant (i.e., Haar) volume form μ_H on the structure group G. Use the map $\Lambda_\gamma : \mathcal{A}_\gamma \to G^E$ defined in (17) to pull-back to $\mathcal{A}_\gamma$ the product volume form $(\mu_\mathrm{H})^E$, induced on G^E by μ_H, to obtain

$$\mu^\mathrm{H}_\gamma := \Lambda_\gamma^{-1}{}_\star\, (\mu_\mathrm{H})^E. \tag{45}$$

We then have the following proposition [5,14,6].

Proposition 5.

(i) *The form μ^H_γ of* (45) *is insensitive to the choice of the gauge over the vertices of γ, used in the definition* (17) *of the map Λ_γ;*

(ii) *the family of volume forms $(\mu^\mathrm{H}_\gamma)_{\gamma\in L}$ satisfies the consistency conditions* (43);

(iii) *the volume form μ_0 defined on $\overline{\mathcal{A}}$ by $(\mu^\mathrm{H}_\gamma)_{\gamma\in L}$ is invariant with respect to the action of (all) the automorphisms of the underlying bundle $B(M, G)$.*

The push forward μ_0' of μ_0, under the natural projection from $\overline{\mathcal{A}}$ to $\overline{\mathcal{A}/\mathcal{G}}$ is the induced Haar measure on $\overline{\mathcal{A}/\mathcal{G}}$ of [5], chronologically, the first measure introduced in this subject. It is invariant with respect to all the diffeomorphisms of M.

The measure μ_0 itself was first introduced in [13]. By now, several infinite families of measures have been introduced on $\overline{\mathcal{A}}$ (which can be pushed forward to $\overline{\mathcal{A}/\mathcal{G}}$) [13,14,9,7]. These are reviewed in [6]. In Section 6, using heat kernel methods, we will introduce another infinite family of measures. These, as well as the measures introduced in [6] arise from C^n volume forms on $\overline{\mathcal{A}}$.

4.3. Vector fields

Introduction of the notion of vector fields on $\overline{\mathcal{A}}$ is somewhat more subtle than that of m-forms because while one can pull back forms, in general one can push forward only vectors (rather than vector fields). Hence, given $\gamma' \geq \gamma$, only certain vector fields on $\mathcal{A}_{\gamma'}$ can be pushed forward through $(p_{\gamma\gamma'})^\star$. To obtain interesting examples, therefore, we now have to introduce an additional structure: vector fields on $\overline{\mathcal{A}}$ will be associated with a graph.

A *smooth vector field* $X^{(\gamma_0)}$ *on* $\overline{\mathcal{A}}$ is a family $(X_\gamma)_{\gamma\geq\gamma_0}$, where X_γ is a smooth vector field on $\mathcal{A}_\gamma$ for all $\gamma \geq \gamma_0$, which satisfies the following consistency condition:

$$(p_{\gamma\gamma'})_\star X_{\gamma'} = X_\gamma, \quad \text{whenever } \gamma' \geq \gamma \geq \gamma_0. \tag{46}$$

It is natural to define a derivation D on $\mathrm{Cyl}^n(\overline{\mathcal{A}})$, as a linear and star preserving map, $D : \mathrm{Cyl}^n(\overline{\mathcal{A}}) \to \mathrm{Cyl}^{n-1}(\overline{\mathcal{A}})$, such that for every $f, g \in \mathrm{Cyl}^n(\overline{\mathcal{A}})$, the Leibniz rule holds, i.e., $D(fg) = D(f)g + D(g)f$. As one might expect, a vector field $X^{(\gamma_0)}$ defines a derivation, which we will denote also by $X^{(\gamma_0)}$. Indeed, given $f \in \mathrm{Cyl}^n(\overline{\mathcal{A}})$ there exists $\gamma \geq \gamma_0$ such that $f = [f_\gamma]_\sim$. We simply set

$$X^{(\gamma_0)}(f) := [X_\gamma(f_\gamma)]_\sim \in \mathrm{Cyl}^{n-1}(\overline{\mathcal{A}}), \tag{47}$$

where $X_\gamma(f_\gamma)$ is the action of the vector field X_γ on $\mathcal{A}_\gamma$ on the function f_γ, and note that the right-hand side is independent of the choice of the representative.

Finally, given any two vector fields, we can take their commutator. We have the following proposition.

Proposition 6. *Let* $X^{(\gamma_1)} = (X_\gamma)_{\gamma\geq\gamma_1}$, $Y^{(\gamma_2)} = (Y_\gamma)_{\gamma\geq\gamma_2}$ *be two vector fields on* $\overline{\mathcal{A}}$. *Then, the commutator* $[X^{(\gamma_1)}, Y^{(\gamma_2)}]$ *of the corresponding derivations is the derivation defined by a vector field* $Z^{(\gamma_3)}$ *on* $\overline{\mathcal{A}}$ *(where* $\gamma_3 \in L$ *is any label satisfying* $\gamma_3 \geq \gamma_1, \gamma_2$*) given by*

$$Z_\gamma = [X_\gamma, Z_\gamma] \tag{48}$$

for any $\gamma \geq \gamma_3$.

For notational simplicity, from now on, we will drop the superscripts on the vector fields.

4.4. Vector fields as momentum operators

We will first introduce the notion of compatibility between vector fields X and volume forms μ on $\overline{\mathcal{A}}$ and then use it to define certain essentially self-adjoint operators $P(X)$ on $L^2(\overline{\mathcal{A}}, \mu)$.

Let us begin by recalling that, given a manifold Σ a vector field V and a volume form ν thereon, the divergence $\mathrm{div}_\nu V$ is a function defined on M by

$$L_V\ \nu\ =:\ (\mathrm{div}_\nu V)\nu, \tag{49}$$

where L_V denotes the standard Lie derivative.

We will say that a vector field $X = (X_\gamma)_{\gamma\geq\gamma_0}$ on $\overline{\mathcal{A}}$ is *compatible with a volume form* $\mu = (\mu_\gamma)_{\gamma\in L}$ on $\overline{\mathcal{A}}$ if

$$p^\star_{\gamma\gamma'}\ \mathrm{div}_{\mu_\gamma} X_\gamma\ =\ \mathrm{div}_{\mu_{\gamma'}} X_{\gamma'}, \tag{50}$$

whenever $\gamma' \geq \gamma \geq \gamma_0$. Note, that if (50) holds, the divergence $\mathrm{div}_{\mu_\gamma} X_\gamma$ is a cylindrical function,

$$\mathrm{div}_\mu\ X := [\mathrm{div}_{\mu_\gamma} X_\gamma]_\sim \in \mathrm{Cyl}^\infty(\overline{\mathcal{A}}). \tag{51}$$

We shall call it *the divergence of* X with respect to a volume form μ. The next proposition shows that the divergence of vector fields on $\overline{\mathcal{A}}$ has several of the properties of the usual, finite-dimensional divergence.

Proposition 7.

(i) *Let X be a vector field and μ, a smooth volume form on $\overline{\mathcal{A}}$ such that X is compatible with μ. Then, for every $f, g \in \mathrm{Cyl}^1(\overline{\mathcal{A}})$,*

$$\int_{\overline{\mathcal{A}}} fX(g)\mu\ =\ -\int_{\overline{\mathcal{A}}} (X(f) + (\mathrm{div}_\mu\ X)f)g\ \mu. \tag{52}$$

(ii) *Suppose that Y is another vector field on $\overline{\mathcal{A}}$ which is compatible with μ. Then, the commutator $[X, Y]$ also is compatible with μ, and*

$$\mathrm{div}_\mu[X, Y]\ =\ X(\mathrm{div}_\mu\ Y)\ -\ Y(\mathrm{div}_\mu\ X). \tag{53}$$

Proof. The result follows immediately by using the properties of the usual divergence of vector fields X_γ and Y_γ on $(\mathcal{A}_\gamma, \mu_\gamma)$ and the consistency conditions satisfied by X_γ, Y_γ and μ_γ. □

We are now ready to introduce the momentum operators. Fix a smooth volume form $\mu = (\mu_\gamma)_{\gamma\in L}$ on $\overline{\mathcal{A}}$. In the Hilbert space $L^2(\overline{\mathcal{A}}, \mu)$, we define below a quantum representation of the Lie algebra of vector fields compatible with μ. Let X be such a vector field on $\overline{\mathcal{A}}$. We assign to X the operator $(P(X), \mathrm{Cyl}^1(\overline{\mathcal{A}}))$ as

$$P(X)\ :=\ \mathrm{i}X\ +\ \tfrac{1}{2}\mathrm{i}(\mathrm{div}_\mu\ X)\,. \tag{54}$$

(Here, $\mathrm{Cyl}^1(\overline{\mathcal{A}})$ is the domain of the operator.) Clearly, $(P(X), \mathrm{Cyl}^1(\overline{\mathcal{A}}))$ is a densely-defined operator on the Hilbert space $L^2(\overline{\mathcal{A}}, \mu)$. Following the terminology used in quantum mechanics on manifolds, we will refer to $P(X)$ as the momentum operator associated with the vector field X. As one might expect from this analogy, the second term in the definition (54) of the momentum operators is necessary to ensure that it is a symmetric operator.

To examine properties of this operator, we first need some general results. Let us therefore make a brief detour and work in a more general setting. Consider a family of Hilbert spaces $(\mathcal{H}_\gamma, p^\star_{\gamma\gamma'})_{\gamma,\gamma'\in\Gamma}$ where Γ is any partially ordered and directed set of labels and

$$p^\star_{\gamma\gamma'} : \mathcal{H}_\gamma \to \mathcal{H}_{\gamma'} \tag{55}$$

is an inner-product preserving embedding defined for each ordered pair $\gamma' \geq \gamma \in \Gamma$. The maps (55) provide the union $\bigcup_{\gamma\in\Gamma} H_\gamma$ with an equivalence relation defined as in (7). The Hermitian inner products $(\cdot,\cdot)_\gamma$ give rise to a unique Hermitian inner product on the vector space $(\bigcup_{\gamma\in\Gamma} H_\gamma)/\sim$. For, if $\psi, \phi \in (\bigcup_{\gamma\in\Gamma} H_\gamma)/\sim$, there exists a common label $\gamma \in \Gamma$ such that $\psi = [\psi_\gamma]_\sim$ and $\phi = [\phi_\gamma]_\sim$, with $\psi_\gamma, \phi_\gamma \in H_\gamma$, and we can set

$$(\psi, \phi) := (\psi_\gamma, \phi_\gamma)_\gamma. \tag{56}$$

It is easy to check that this inner product is Hermitian. Thus, we have a pre-Hilbert space. Let $\mathcal{H}$ denote its Cauchy completion:

$$\mathcal{H} = \overline{\bigcup_{\gamma\in\Gamma} \mathcal{H}_\gamma/\sim}. \tag{57}$$

On this Hilbert space $\mathcal{H}$, consider an operator given by a family of operators $(O_\gamma, \mathcal{D}_\gamma(O_\gamma))_{\gamma\in\Gamma(O)}$, where $\Gamma(O) \subset \Gamma$ is a cofinal subset of labels (i.e., for every $\gamma \in \Gamma$ there is $\gamma' \in \Gamma(O)$ such that $\gamma' \geq \gamma$). We will say that $(O_\gamma, \mathcal{D}_\gamma(O_\gamma))_{\gamma\in\Gamma(O)}$ is self-consistent if the following two conditions are satisfied:

$$p^\star_{\gamma\gamma'}\mathcal{D}_\gamma(O_\gamma) \subseteq \mathcal{D}_{\gamma'}(O_{\gamma'}), \tag{58}$$

$$O_{\gamma'} \circ p^\star_{\gamma\gamma'} = p^\star_{\gamma\gamma'} \circ O_\gamma \tag{59}$$

for every $\gamma' \geq \gamma$ such that $\gamma', \gamma \in \Gamma(O)$. Since the label set $\Gamma(O) \subset \Gamma$ is cofinal, a self-consistent family of operators $(O_\gamma, \mathcal{D}_\gamma(O_\gamma))_{\gamma\in\Gamma(\mathcal{P})}$ defines an operator O in $\mathcal{H}$ via $O(\psi) := [O_\gamma\psi_\gamma]_\sim$.

A general result which we will apply to the momentum operators is the following.

Lemma 1. *Let $(O_\gamma, \mathcal{D}_\gamma(O_\gamma))_{\gamma\in\Gamma(O)}$ be a self-consistent family of operators and $\Gamma(O)$ be cofinal in Γ. Then:*

(i) *$(O_\gamma, \mathcal{D}_\gamma(O_\gamma))_{\gamma\in\Gamma(\mathcal{P})}$ defines uniquely an operator O in $\mathcal{H}$ acting on a domain $\mathcal{D}(O) := \bigcup_{\gamma\in\Gamma(O)} \mathcal{D}_\gamma(O_\gamma)/\sim$ and such that for every $f_\gamma \in \mathcal{D}_\gamma(O_\gamma)$*

$$O([f_\gamma]_\sim) = [O_\gamma(f_\gamma)]_\sim; \tag{60}$$

(ii) *if $(O_\gamma, \mathcal{D}_\gamma(O_\gamma))$ is essentially self-adjoint in H_γ for every $\gamma \in \Gamma(O)$, then the resulting operator $(O, \mathcal{D}(O))$ defined in (i) is also essentially self-adjoint;*

(iii) *if* $(O_\gamma, \mathcal{D}_\gamma(O_\gamma))$ *is essentially self-adjoint in* H_γ *for every* $\gamma \in \Gamma(O)$, *then the family of the self-dual extensions* $(\tilde{O}_\gamma, \mathcal{D}_\gamma(\tilde{O}_\gamma))_{\gamma\in\Gamma(O)}$ *is self-consistent.*

Proof. Part (i) is obvious from the above discussion.

We will prove (ii) by showing that the ranges of the operators $O + \mathrm{i}I$ and $O - \mathrm{i}I$, where I is the identity operator, are dense in $\mathcal{H}$. They are given by

$$(O \pm \mathrm{i}I)(\mathcal{D}(O)) = \bigcup_{\gamma\in\Gamma(O)} (O_\gamma \pm \mathrm{i}I)(\mathcal{D}_\gamma(O_\gamma))/\sim . \tag{61}$$

But, as follows from the hypothesis, the range of each of the operators $O_\gamma \pm \mathrm{i}I$ is dense in the corresponding H_γ. Hence indeed, the right-hand side of (61) is dense in $\mathcal{H}$.

To show (iii), recall that the self-adjoint extension of an essentially self-adjoint operator is just its closure. Let $\gamma', \gamma \in \Gamma(O)$ and $\gamma' \geq \gamma$. Via the pull-back $p^\star_{\gamma\gamma'}$, we may consider H_γ as a subspace of $\mathcal{H}_{\gamma'}$. Since $(O_{\gamma'}, \mathcal{D}_{\gamma'}(O_{\gamma'}))$ is an extension of $(O_\gamma, \mathcal{D}_\gamma(O_\gamma))$, the closure $(\tilde{O}_{\gamma'}, \mathcal{D}_{\gamma'}(\tilde{O}_{\gamma'}))$ is still an extension for $(\tilde{O}_\gamma, \mathcal{D}_\gamma(\tilde{O}_\gamma))$. This concludes the proof of the lemma. □

We can now return to the momentum operators $(P(X), \mathrm{Cyl}^1(\overline{\mathcal{A}}))$ on $L^2(\overline{\mathcal{A}}, \mu)$.

Theorem 2.

(i) *Let* $X = (X_\gamma)_{\gamma\geq\gamma_0}$ *and* $\mu = (\mu_\gamma)_{\gamma\in L}$ *be a smooth vector field and volume form on the projective limit* $\overline{\mathcal{A}}$. *Suppose* X *is compatible with* μ; *then, the operator* $(P(Y), \mathrm{Cyl}^1(\overline{\mathcal{A}}))$ *of* (54) *is essentially self-adjoint on* $L^2(\overline{\mathcal{A}}, \mu)$;

(ii) *let* Y *be another smooth vector field on the projective limit, also compatible with the measure* μ. *Then, the vector field* $[X, Y]$ *also is compatible with* μ *and*

$$P([X, Y]) = \mathrm{i}[P(X), P(Y)]. \tag{62}$$

Proof. Part (i) of the theorem follows trivially from Lemma 1; we only have to substitute $L^2(\overline{\mathcal{A}}, \mu)$ for $\mathcal{H}$, $L^2(\mathcal{A}_\gamma, \mu_\gamma)$ for H_γ, L for Γ and $((\mathrm{i}(Y_\gamma + \frac{1}{2}\mathrm{div}_{\mu_\gamma} Y_\gamma))_{\gamma\geq\gamma_0}, \mathrm{Cyl}^1(\overline{\mathcal{A}}))$ for $(O_\gamma, \mathcal{D}_\gamma(O_\gamma))_{\gamma\in\Gamma(O)}$.

Finally, part (ii) can be shown by a simple calculation using Proposition 7. □

This concludes our discussion of the momentum operators. Most of the results of this section concern the case when a vector field is compatible with a volume form. In the next section we shall see that a natural symmetry condition implies that a vector field on $\overline{\mathcal{A}}$ is necessarily compatible with the Haar volume form.

5. Elements of differential geometry on $\overline{\mathcal{A}/\mathcal{G}}$

We now turn to $\overline{\mathcal{A}/\mathcal{G}}$, the space that we are directly interested in.

5.1. Forms and volume forms

Let us begin with functions. We know from Section 3.4 that

$$\overline{\mathcal{A}/\mathcal{G}} = \overline{\mathcal{A}}/\overline{\mathcal{G}} . \tag{63}$$

Therefore, we can drop the distinction between functions on $\overline{\mathcal{A}/\mathcal{G}}$ and $\overline{\mathcal{G}}$-invariant functions defined on $\overline{\mathcal{A}}$. In particular, we can identify the $\mathbb{C}^\star$-algebra $\mathbb{C}^0(\overline{\mathcal{A}/\mathcal{G}})$ of continuous functions on $\overline{\mathcal{A}/\mathcal{G}}$ with the $\mathbb{C}^\star$-subalgebra of $\overline{\mathcal{G}}$- invariant elements of $\mathbb{C}^0(\overline{\mathcal{A}})$. This suggests that we adopt the same strategy towards differentiable functions and forms. Therefore, we will let $\mathrm{Cyl}^n(\overline{\mathcal{A}/\mathcal{G}})$ be the $\star$-subalgebra of $\overline{\mathcal{G}}$-invariant elements of $\mathrm{Cyl}^n(\mathcal{A})$, and $\Omega(\overline{\mathcal{A}/\mathcal{G}})$ be the subalgebra consisting of $\mathcal{G}$-invariant elements of Grassmann algebra $\Omega(\overline{\mathcal{A}})$. The operations of taking exterior product and exterior derivative is well defined on $\Omega(\overline{\mathcal{A}/\mathcal{G}})$.

Similarly, by a volume form on $\overline{\mathcal{A}/\mathcal{G}}$, we shall mean a $\mathcal{G}$-invariant volume form on $\overline{\mathcal{A}}$. As noted in Section 4.2, the induced Haar form on $\overline{\mathcal{A}}$ is $\overline{\mathcal{G}}$ invariant and provides us with a natural measure on $\overline{\mathcal{A}/\mathcal{G}}$. Furthermore, since $\overline{\mathcal{G}}$ is compact, we can extract the $\overline{\mathcal{G}}$-invariant part of *any* volume form ν on $\overline{\mathcal{A}}$ by an averaging procedure (see also [13]).

Proposition 8. *Suppose $\nu = (\nu_\gamma)_{\gamma\in L}$ is a volume form on $\overline{\mathcal{A}}$. Then, if $R_{\bar g}$ denotes the action of $\bar g \in \overline{\mathcal{G}}$ on $\overline{\mathcal{A}}$, and* $\mathrm{d}\bar g$ *denotes the Haar measure on $\overline{\mathcal{G}}$,*

$$\bar\nu := \int_{\overline{\mathcal{G}}} (R_{\bar g})_\star \nu \, \mathrm{d}\bar g \tag{64}$$

is a $\overline{\mathcal{G}}$-invariant volume form on $\overline{\mathcal{A}}$ such that for every $\overline{\mathcal{G}}$-invariant function $f \in \mathbb{C}^0(\overline{\mathcal{A}})$

$$\int_{\overline{\mathcal{A}}} f\,\nu = \int_{\overline{\mathcal{A}}} f\,\bar\nu . \tag{65}$$

In terms of the projective family, we can write out the averaged volume form more explicitly. Let $\nu = (\nu_\gamma)_{\gamma\in L}$. Then, the averaged volume-form $\bar\nu = (\overline{\nu_\gamma})_{\gamma\in L}$ where

$$\overline{\nu_\gamma} = \int_{G^V} (R_{(g_1,\ldots,g_V)})_\star \nu_\gamma \mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_V , \tag{66}$$

where $R_{(g_1,\ldots,g_V)}$ denotes the action of $(g_1,\ldots,g_V) \in \mathcal{G}_\gamma$ on $\mathcal{A}_\gamma$ ($\mathcal{G}_\gamma$ being identified with G^V).

5.2. Vector fields on $\overline{\mathcal{A}/\mathcal{G}}$

The procedure that led us to forms on $\overline{\mathcal{A}/\mathcal{G}}$ can also be used to define vector fields on $\overline{\mathcal{A}/\mathcal{G}}$. Furthermore, for vector fields, one can obtain some general results which are directly useful in defining quantum operators. We will establish these results in this subsection and use them to obtain a complete characterization of vector fields on $\overline{\mathcal{A}/\mathcal{G}}$ in the next subsection.

The group $\overline{\mathcal{G}}$ acts on vector fields on $\overline{\mathcal{A}}$ as follows. Let $X = (X_\gamma)_{\gamma\geq\gamma_0}$, and $\bar{g} = (g_\gamma)_{\gamma\in L} \in \overline{\mathcal{G}}$. Then

$$R_{\bar{g}} \star X \ := \ (R_{g_\gamma \star}\, X_\gamma)_{\gamma\geq\gamma_0}\,, \tag{67}$$

where, as before, $R_{g_\gamma} \ : \mathcal{A}_\gamma \ \to \ \mathcal{A}_\gamma$ is the action of $\mathcal{G}_\gamma$ on $\mathcal{A}_\gamma$. In the first part of this subsection, we will explore the relation between $\overline{\mathcal{G}}$-invariant vector fields and the induced Haar form μ_0 on $\overline{\mathcal{A}}$. Since we are dealing here only with the Haar measure, for simplicity of notation, in this subsection, we will drop the (measure-)suffix on divergence.

Theorem 3. *Let $X = (X_\gamma)_{\gamma\geq\gamma_0}$ be a $\mathbb{C}^{n+1}$ vector field on $\overline{\mathcal{A}}$. If X is $\overline{\mathcal{G}}$-invariant then it is compatible with the Haar measure μ_0 on $\overline{\mathcal{A}}$ and* div $X \in \mathrm{Cyl}^n(\overline{\mathcal{A}})$ *is $\overline{\mathcal{G}}$-invariant.*

Proof. We need to show that, if $\gamma, \gamma' \geq \gamma_0$, then

$$\mathrm{div}\ X_\gamma \sim \mathrm{div}\ X_{\gamma'}\,. \tag{68}$$

Since the family $(X_\gamma)_{\gamma\geq\gamma_0}$ is consistent, it is sufficient to show that if $\gamma_2 \geq \gamma_1$, then

$$p^\star_{\gamma_1\gamma_2}\,(\mathrm{div}(p_{\gamma_1\gamma_2})_\star\, X_{\gamma_2}) = \mathrm{div}\ X_{\gamma_2}\,. \tag{69}$$

Now, the graph γ_2 consists of say two types of edges: (i) edges $e_1, \ldots, e_{E_3}$, which are contained in γ_1 and (ii) the remaining edges $e'_{E_3+1}, \ldots, e'_{E_2}$. The first set of edges forms a graph $\gamma_3 \geq \gamma_1$ whose image coincides with that of γ_1. Therefore, in particular, we have $\gamma_3 \geq \gamma_0$ and X_{γ_3} is well defined. Our strategy is to decompose the projection $p_{\gamma_1\gamma_2}$ as

$$p_{\gamma_1\gamma_2} \ = \ p_{\gamma_1\gamma_3} \circ p_{\gamma_3\gamma_2} \tag{70}$$

and prove that each of the two projections on the right-hand side satisfies (69), i.e., that we have

$$p^\star_{\gamma_1\gamma_3}\,(\mathrm{div}(p_{\gamma_1\gamma_3})_\star\, X_{\gamma_3}) = \mathrm{div}\ X_{\gamma_3} \tag{71}$$

and

$$p^\star_{\gamma_3\gamma_2}\,(\mathrm{div}(p_{\gamma_3\gamma_2})_\star\, X_{\gamma_2}) = \mathrm{div}\ X_{\gamma_2}\,. \tag{72}$$

These two results will be established in two lemmas which will conclude the proof of the main part of the theorem.

Once this part is established, the $\overline{\mathcal{G}}$-invariance of $\mathrm{div}(X) \in \mathrm{Cyl}^\infty(\overline{\mathcal{A}})$ is obvious from the $\overline{\mathcal{G}}$-invariance of the vector field X and of the measure μ_0.

Lemma 2. *Let $\gamma_3 \geq \gamma_1$ be such that the images of γ_3 and γ_1 in § coincide. Let X_{γ_3} be a $\mathcal{G}_{\gamma_3}$-invariant vector field on $\mathcal{A}_{\gamma_3}$ and X_{γ_1}, a $\mathcal{G}_{\gamma_1}$-invariant vector field on A_{γ_1} such that $(p_{\gamma_1\gamma_3})_\star X_{\gamma_3} \ = \ X_{\gamma_1}$. Then,*

$$p^\star_{\gamma_1\gamma_3}(\mathrm{div}\ X_{\gamma_1}) \ = \ \mathrm{div}\ X_{\gamma_3}. \tag{73}$$

Proof. Since γ_3 is obtained just by subdividing some of the edges of γ_1, it follows that the pull-back $p^\star_{\gamma_1\gamma_3}$ is an isomorphism of the $\mathbb{C}^\star$-algebra of continuous and $\mathcal{G}_{\gamma_1}$- invariant functions on $\mathcal{A}_{\gamma_1}$ into the $C^\star$-algebra of continuous and $\mathcal{G}_{\gamma_3}$-invariant functions on $\mathcal{A}_{\gamma_3}$. Hence it defines an isomorphism between the corresponding Hilbert spaces. The vector fields X_{γ_1} and X_{γ_3} define the operators which are equal to each other via the isomorphism. Hence, their divergences are also equivalent as operators, and being smooth functions, are just equal to each other, modulo the pull-back.

Now, we turn to the second part, (72), of the proof.

Lemma 3. *Let $X = (X_\gamma)_{\gamma\geq\gamma_0}$ be a $\mathcal{G}$-invariant vector field on $\overline{\mathcal{A}}$. Let $\gamma_2 \geq \gamma_3 \geq \gamma_0$ be such that γ_2 is obtained by adding edges to γ_3 the images of all of which, except possibly the end points, lie outside the image of γ_3. Then,*

$$p^\star_{\gamma_3\gamma_2}(\mathrm{div}\ X_{\gamma_3}) = \mathrm{div}\ X_{\gamma_2}\ . \tag{74}$$

Proof. Via appropriate parametrization we can set

$$\mathcal{A}_{\gamma_2} = \mathcal{A}_{\gamma_3} \times G^{E_2-E_3}, \tag{75}$$

so that the map $p_{\gamma_3\gamma_2}$, becomes the obvious projection

$$p_{\gamma_3\gamma_2} : \mathcal{A}_{\gamma_3} \times G^{E_2-E_3} \to \mathcal{A}_{\gamma_3}\ . \tag{76}$$

Since X_{γ_2} projects unambiguously to X_{γ_3}, it follows that we decompose X_{γ_2} as

$$X_{\gamma_2} = (X_{\gamma_3}, X_{E_3+1}, \ldots, X_{E_2}), \tag{77}$$

where, for each choice of a point on $\mathcal{A}_{\gamma_3}$, and of variables g_{E_3+j}, $j \neq i$, we can regard X_{E_3+i}, as a vector field on G. (Here, $i = 1, \ldots, E_2 - E_3$, and $\mathcal{A}_{E_3+k}$ is identified with G.)

We will now analyze the properties of these vector fields. Let us fix an edge e_{E_3+i}. We will now show that X_{E_3+i} does not change as we vary $g_{E_3+1}, \ldots, g_{E_3+i-1}, g_{E_3+i+1}, \ldots, g_{E_2}$. Let us suppose that there exist some edges e_{E_3+j} which are *removable* in the sense that one can obtain a closed graph $\gamma_4 \geq \gamma_3$ after removing them. Clearly, $\gamma_2 \geq \gamma_4 \geq \gamma_0$. Hence, there is a vector field X_{γ_4} on $\mathcal{A}_{\gamma_4}$ such that $X_{\gamma_3}, X_{\gamma_4}, X_{\gamma_2}$ are all consistent. This implies that X_{E_3+i} does not change if we vary g_{E_3+j}. Now let us consider the case when edges of γ_3 are not removable. Then, we can construct a closed graph γ_5

$$\gamma_5 := \gamma_2 \cup \{e_+, e_-\} \tag{78}$$

by adding two new edges $e_\pm$ to join the vertices of e_{E_3+i} to any two vertices of γ_3. Then, $\gamma_5 \geq \gamma_2 \geq \gamma_3$ and we have consistency of the vector fields $X_{\gamma_5}, X_{\gamma_2}, X_{\gamma_3}$. Clearly, the X_{E_3+i} component of X_{γ_5} coincides with the X_{E_3+i} component of X_{γ_2}. But in X_{γ_5}, all the edges e_{E_3+j} with $j \neq i$ are removable. Hence, X_{E_3+i} does not depend on g_{E_3+j} if $i \neq j$. Thus, we have shown that X_{E_3+i} is independent of g_j if $i \neq j$.

So far, in this lemma, we have only used the consistency of $(X_\gamma)_{\gamma\geq\gamma_0}$. We now use the $\mathcal{G}$-invariance of X to show that (for each $g_{\gamma_3} \in \mathcal{A}_{\gamma_3}$) X_{E_3+i} is a left-invariant vector field on G. Let v be a vertex of e_{E_3+i} which is not contained in γ_3. For definiteness, let us suppose that it is the final vertex. Then, under the gauge transformation a in the fiber over this vertex, we have

$$(a_\star X_{\gamma_2})_{E+i} = (L_{a^{-1}})_\star X_{E+i}\,, \tag{79}$$

where the left-hand side is the E_{3+i}th component of the vector field in the parenthesis and L_a is the left action of a on G. (Here, we have used our earlier result that X_{E+i} does not depend on g_{E+j} when $i \neq j$.) Now, from $\mathcal{G}_{\gamma_2}$-invariance of X_{γ_2}, it follows that

$$X_{E+i} = L_{a\star} X_{E+i}. \tag{80}$$

This conclusion applies to any value of $i = 1, \ldots, E_2 - E_3$. Thus, for each choice of $g_{\gamma_3} \in \mathcal{A}_{\gamma_3}$, X_{E_3+i} are, in particular, divergence-free vector fields on G.

We now collect these results to compute the divergence of $\mathcal{X}_{\gamma_2}$:

$$\mathrm{div} X_{\gamma_2} = \mathrm{div} X_{\gamma_3} + \mathrm{div} X_{E_3+1} + \cdots + \mathrm{div} X_{E_2} = \mathrm{div} X_{\gamma_3}\,, \tag{81}$$

where, for simplicity of notation, we have dropped the pull-back symbols. □

Using this result and those of Section 4.4, we have the following theorem on the operators on $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$ defined by $\overline{\mathcal{G}}$-invariant vector fields on $\overline{\mathcal{A}}$.

Theorem 4. *Let X be a $\overline{\mathcal{G}}$-invariant vector field on $\overline{\mathcal{A}}$. The operator*

$$P(X) := \mathrm{i}(X + \tfrac{1}{2}\mathrm{div} X) \tag{82}$$

with domain $\mathrm{Cyl}^1(\overline{\mathcal{A}/\mathcal{G}})$ *is essentially self-adjoint on $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$. Suppose Y is another $\overline{\mathcal{G}}$-invariant vector field on $\overline{\mathcal{A}/\mathcal{G}}$; then, $[X, Y]$ is also a $\overline{\mathcal{G}}$-invariant vector field on $\overline{\mathcal{A}/\mathcal{G}}$ and*

$$P([X, Y]) = \mathrm{i}[P(X), P(Y)] \tag{83}$$

on $\mathrm{Cyl}^2(\overline{\mathcal{A}/\mathcal{G}})$.

5.3. Characterization of vector fields on $\overline{\mathcal{A}/\mathcal{G}}$

In the previous subsection we showed that the $\overline{\mathcal{G}}$-invariant vector fields on $\overline{\mathcal{A}}$ have interesting properties. It is therefore of considerable interest to have control on the structure of such vector fields. Can one construct them explicitly? What is the available freedom? To answer such questions, we will now obtain a complete characterization of the $\overline{\mathcal{G}}$-invariant vector fields on $\overline{\mathcal{A}}$ in the case when G is assumed to be semisimple.

Fix a graph γ_0. To construct a $\overline{\mathcal{G}}$-invariant vector field $X = (X_\gamma)_{\gamma \geq \gamma_0}$ on $\overline{\mathcal{A}}$, we have, first of all, to specify a $\mathcal{G}_{\gamma_0}$-invariant vector field on $\mathcal{A}_{\gamma_0}$. We want to analyze the freedom available in extending this vector field to $\mathcal{A}_\gamma$ for all $\gamma \geq \gamma_0$. Now the edges of any $\gamma \geq \gamma_0$ can be ordered in such a way that:

(1) the first n edges, $e_1, \ldots, e_n$, are contained in γ_0, for an appropriate n;
(2) the next $m - n$ edges, $e_{n+1}, \ldots, e_m$, begin at (i.e., have one of their vertices) on γ_0, for some m; and
(3) the remaining edges, say, $e_{m+1}, \ldots, e_k$, do not intersect γ_0 at all.

Hence, we can decompose $\mathcal{A}_\gamma$ as

$$\mathcal{A}_\gamma = \mathcal{A}_{\gamma_1} \times G^{m-n} \times G^{k-m}\,, \tag{84}$$

where $\gamma_1 \geq \gamma_0$ is the graph formed by the first n edges. Given X_{γ_0}, the projection $\mathcal{A}_{\gamma_1} \to \mathcal{A}_{\gamma_0}$ determines – via consistency conditions – the vector X_{γ_1} modulo a vector field tangent to the fibers of the projection. Next, consider the last $k-m$ components of X_γ corresponding to the last set of edges. Since both vertices of these edges lie outside γ_0, the corresponding vector fields on G have to be both right and left invariant. Since we have assumed that the gauge group is semisimple, this implies that they must vanish. Thus, the essential freedom in extending X_{γ_0} to X_γ lies in the $m-n$ components of X_γ associated with the second set of edges. Hence, using the notation of Lemma 3, we can express X_γ as

$$X_\gamma := (X_{\gamma_1}, F([e_{n+1}]), \dots, F([e_m]), 0, \dots, 0)\,. \tag{85}$$

Here, $F([e_i])$ with $i = n+1, \dots, m$ is a function

$$F([e_i]) : \mathcal{A}_{\gamma_1} \to \Gamma(T(\mathcal{A}_{e_i}))$$

whose values are $\mathcal{G}_{v_i}$-invariant vector fields on $\mathcal{A}_{e_i}$, where v_i is the end of the edge e_i which is not contained in γ_0. The $\overline{\mathcal{G}}$-invariance of X implies that $F([e_i])$ should have certain transformation properties under the action of the groups $\mathcal{G}_v$ which act on the fibers over the vertices v of γ_0. Given $a_v \in \mathcal{G}_v$, we need: $F([e_i]) \circ a_v = (a_v)_\star^{-1} F([e_i])$ if v is the vertex of e_i and $F([e_i]) \circ a_v = F([e_i])$ otherwise.

We can now summarize the information that is necessary and sufficient to define a $\overline{\mathcal{G}}$-invariant vector field X on $\overline{\mathcal{A}}$. First, we need a graph γ_0. For each $x \in \gamma_0$, let e_x be the set of germs of edges (i.e., the data at x that is necessary and sufficient to specify edges) which do not overlap with any of the edges of γ_0 that pass through x. (Recall that edges are all analytic.) Let $\mathcal{P}_{\gamma_0}$ be the sheaf of germs of transversal edges over the γ_0 divided by the reparametrizations, i.e., set

$$\mathcal{P}_{\gamma_0} := \bigcup_{x\in\gamma_0} \mathcal{P}_x\,. \tag{86}$$

Next, given a point x on γ_0 let $\gamma_x \geq \gamma_0$ be the graph obtained by cutting the edge on which x lies into two at x. (If x is a vertex of γ_0, then $\gamma_x = \gamma_0$.) Finally, choose a point in the fiber of the underlying bundle $B(M, G)$ over each point of every edge on γ_0. Up to this freedom, the group $\mathcal{G}_x$ is then identified with G. Then, the necessary and sufficient data for constructing a $\overline{\mathcal{G}}$-invariant vector field $X = (X_\gamma)_{\gamma\geq\gamma_0}$ (regarded as an operator on $\overline{\mathcal{G}}$-invariant function on $\overline{\mathcal{A}}$) are the following:

(1) A $\mathcal{G}_{\gamma_0}$-invariant vector field X_{γ_0} on $\mathcal{A}_{\gamma_0}$ and a $\mathcal{G}_{\gamma_1}$ - invariant Vector Field X_{γ_1} on $\mathcal{A}_{\gamma_1}$ consistent with X_{γ_0}, for every γ_1 obtained from γ_0 simply by subdividing edges;
(2) A map from the set of germs of transversal edges $\mathcal{P}_{\gamma_0}$ into the Lie algebra (of left-invariant vector fields on G) LG-valued functions on $\mathcal{A}_{\gamma_x}$,

$$F : \mathcal{P}_{\gamma_0} \to \mathbb{C}^n(\mathcal{A}_{\gamma_x}) \otimes LG; \tag{87}$$

which has the following transformation properties with respect to the group $\mathcal{G}_{\gamma_x}$:

$$F([e]_x) \circ a_v = \begin{cases} F([e]_x) & \text{if } v \neq x \\ a^{-1} F([e]_x) a & \text{if } v = x \end{cases} \tag{88}$$

for every vertex v of γ_x. Then, given an edge e intersecting γ at x and the corresponding space $\mathcal{A}_e$, a value of $F([e])$ defines unambiguously a $\mathcal{G}_{v'}$-invariant vector field on $\mathcal{A}_e$, v' standing for the other end of e (because the remaining freedom of a gauge for $\mathcal{A}_e$ is covered by the action of $\mathcal{G}_{v'}$).

Thus, there is a rich variety of $\overline{\mathcal{G}}$-invariant vector fields on $\overline{\mathcal{A}}$. The vector field is $\overline{\mathcal{G}}$-invariant. If the cometric tensor and the 1-form are $\overline{\mathcal{G}}$-invariant, so is the vector field.

We will conclude this discussion of vector fields by pointing out that, if one is interested only in the action of the vector fields on cylindrical functions, a priori there appears to be some freedom in one's choice of the initial definition itself. There are at least three ways of modifying the definition we used.

(1) First, we could have chosen another set of labels. Our definition used graphs $\gamma \geq \gamma_0$ for some γ_0. Instead, we could have labelled the vector fields by *any* cofinal L. However, it is not clear if all our results would go through in this more general setting. In particular, it is not obvious that the vector fields would then form a Lie algebra.

(2) Another possibility is to use the same labels ($\gamma \geq \gamma_0$ for some γ_0) but to weaken the consistency conditions slightly. Since we only want to act these vector fields $(X_\gamma)_{\gamma\geq\gamma_0}$ on functions which are $\overline{\mathcal{G}}$-invariant, it would suffice to require only that the consistency conditions are satisfied "modulo the gauge directions". That is, one might require only that each X_γ is $\mathcal{G}_\gamma$-invariant and $p_{\gamma\gamma'\star}X_{\gamma'} = X_\gamma$ if $\gamma' \geq \gamma$, *both modulo the directions tangent to the fibers of the group* $\mathcal{G}_\gamma$. However, then it is no longer clear that the notion of divergence of X is well defined. Further work is needed.

(3) Finally, throughout this paper, we have considered projective families labelled by graphs. Alternatively, one can also consider projective families labelled by subgroups of the group of equivalence classes of closed, based loops in M, where two loops are equivalent if the holonomy of any connection around them, evaluated at the base point, is the same (see [21,6]). One can define $\overline{\mathcal{G}}$-invariant vector fields in this setting as well and the resulting momentum operators on $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$ are essentially the same as those introduced here. However, the proofs are more complicated since they essentially involve decomposing loops in to graphs used here.

6. Laplacians, heat equations and heat kernel measures on $\overline{\mathcal{A}/\mathcal{G}}$ associated with edge-metrics

In the last two sections, we saw that, although $\overline{\mathcal{A}}$ and $\overline{\mathcal{A}/\mathcal{G}}$ initially arise only as compact topological spaces, using graphs on M and the geometry of the Lie group G one can introduce on them, quite successfully, structures normally associated with manifolds. Therefore, a natural question now arises: Can one exploit the invariant Riemannian geometry on G to define on $\overline{\mathcal{A}/\mathcal{G}}$ new structures? In this section, we will show that the answer is in the affirmative.

6.1. A Laplace operator

Let us fix an (left and right) invariant metric tensor k on G. The obvious strategy – which, e.g., successfully led us to the Haar volume form in Section 4.2 – would be to use the fact that $\mathcal{A}_\gamma$ can be identified with G^E, to endow it with the product metric k'_γ and let Δ'_γ be the associated Laplacian operator. (Here, as before, E is the number of edges in the graph γ.) Unfortunately, this strategy fails: the resulting family of operators $(\Delta'_\gamma)_{\gamma\in L}$ fails to be self-consistent. (This is why we have used the prime in k' and Δ'.) This is a good illustration of the subtlety of the consistency conditions and brings out the nontriviality of the fact that the families that led us to forms, vector fields and the Haar measure turned out to be consistent.

The "minimal" modification that leads to a Laplacian requires an additional ingredient: a metric on the space of edges on M. An *edge-metric on M*, will be a map which assigns to each edge (i.e., finite, analytic curve) e in M a nonnegative number, $l(e)$ which is independent of the orientation of e and additive, i.e. satisfies

$$l(e^{-1}) = l(e) \quad \text{and} \quad l(e_1 \circ e_2) = l(e_1) + l(e_2)\,. \tag{89}$$

l can be thought of as a generalized "length" function on the space of edges. The technique of using such "an additive weight" was suggested by certain methods employed by Kondracki and Klimek [16] in the context of two-dimensional Yang–Mills theory.

It is not difficult to construct edge-metrics explicitly. Two simple examples of such constructions are:

(1) Introduce a Riemannian metric g on M and let $l(e)$ be the length of e.

(2) Fix a collection s of analytic surfaces in M and define $l(e)$ to be the number of isolated points of intersection between e and s.

Given an edge-metric l, for each graph γ we define on $\mathcal{A}_\gamma(= G^E)$ the following "weighted" Laplacian

$$\Delta_{\gamma,(l)} := l(e_1)\Delta_{e_1} + \cdots + l(e_E)\Delta_{e_E}\,, \tag{90}$$

where Δ_{e_i} is an operator which applied to a function $f_\gamma(g_{e_1}, \ldots, g_{e_E})$ acts (only) on the G-variable g_{e_i} as the Laplacian of the metric k on G. It is not obvious that this $\Delta_{\gamma,(l)}$ is well defined since the isomorphism between $\mathcal{A}_\gamma$ by G^E used in the above construction is not unique. However, two such isomorphisms are in essence related by an element of $\mathcal{G}_\gamma$ and since Δ is left and right invariant on G, $\Delta_{\gamma,(l)}$ is well defined; it is insensitive to the ambiguity in the choice of $(g_{e_1}, \ldots, g_{e_E})$ that label the points of $\mathcal{A}_\gamma$. Furthermore, we have the following theorem.

Theorem 5.

(i) *The family of operators* $(\Delta_{\gamma,(l)}, \mathbb{C}^2(\mathcal{A}_\gamma))_{\gamma\in L}$ *is self-consistent (in the sense of* (58) *and* (59));

(ii) *The operator* $(\Delta_{(l)}, \mathrm{Cyl}^2(\overline{\mathcal{A}}))$ *defined in the Hilbert space* $L^2(\overline{\mathcal{A}}, \mu_0)$, *where* μ_0 *is the Haar volume form on* $\overline{\mathcal{A}}$, *by*

$$\Delta_{(l)}\left([f_\gamma]_\sim\right) := [\Delta_{\gamma,(l)}(f_\gamma)]_\sim \tag{91}$$

is essentially self-adjoint;

(iii) *the operator $\Delta_{(l)}$ is $\overline{\mathcal{G}}$-invariant and thus defines an essentially self-adjoint operator $(\Delta_{(l)}, \mathrm{Cyl}^2(\overline{\mathcal{A}/\mathcal{G}}))$ in $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0')$, where μ_0' is the push forward of μ_0 to $\overline{\mathcal{A}/\mathcal{G}}$.*

Proof. Note first that, for every $\gamma' \geq \gamma$, the projection $p_{\gamma\gamma'} : \mathcal{A}_{\gamma'} \to \mathcal{A}_\gamma$ can be written as a composition of projections

$$p_{\gamma\gamma'} = p_{\gamma\gamma_1} \circ \cdots \circ p_{\gamma_n\gamma'} \tag{92}$$

each of the terms being one of the following three kinds:

$$(g_1, \ldots, g_k, \ldots) \mapsto (g_1, \ldots, (g_k)^{-1}, \ldots),$$
$$(g_1, \ldots, g_{k-1}, g_k, \ldots) \mapsto (g_1, \ldots, g_{k-1}, \ldots), \tag{93}$$
$$(g_1, \ldots, g_{k-1}, g_k) \mapsto (g_1, \ldots, g_{k-1}g_k). \tag{94}$$

The operators $\Delta_{l(e),\gamma}$ are automatically consistent with projections of the second class above; no conditions need to be imposed on $l(e_i)$. However, to be consistent with a projection of the third kind, the numbers $l(e)$ have to satisfy the following necessary and sufficient condition. Let f be a function on G. Whenever we divide an edge e into $e = e_2 \circ e_1$, then

$$(l(e_1)\Delta_{e_1} + l(e_2)\Delta_{e_2}) f(g_2, g_1) = (l(e)\Delta f)(g_2 g_1), \tag{95}$$

where $f(g_2, g_1) = f(g_2 g_1)$. A short calculation shows that the necessary condition for this to hold is precisely our second restriction on $l(e)$: $l(e) = l(e_1) + l(e_2)$. Similarly, our first restriction $l(e^{-1}) = l(e)$ is necessary and sufficient to ensure consistency with respect to the projections of the first type. This establishes part (i) of the theorem.

Part (ii) follows easily from Lemma 1 and from the essential self-adjointness of the operators $(\Delta_\gamma^{(l)}, \mathrm{C}^2(\mathcal{A}_\gamma))$ defined in $L^2(\mathcal{A}_\gamma, (\mu_{\mathrm{H}})^E)$. Part (iii) is essentially obvious. □

Remark 6. If $l(e)$ is nonzero for every nontrivial edge – as is the case for the first example of $l(e)$ – we can introduce on each $\mathcal{A}_\gamma = G^E$ a block diagonal metric tensor $k_\gamma := (l(e_1)^{-1}k, \ldots, l(e_E)^{-1}k)$. The operator $\Delta_\gamma^{(l)}$ is just the Laplacian on $\mathcal{A}_\gamma$ defined by k_γ.

6.2. *The heat equation and the associated volume forms*

Given an edge metric $l(e)$, we have a Laplacian on $\overline{\mathcal{A}}$. It is now natural to use these Laplacians to formulate heat equations on $\overline{\mathcal{A}}$, to find the corresponding heat kernels and to construct the associated heat-kernel measures on $\overline{\mathcal{A}}$. These would be natural analogs of the usual Gaussian measures on topological vector spaces.

As usual, heat equations will involve a parameter t with $0 < t$ and a Laplacian $\Delta_{(l)}$ on $\overline{\mathcal{A}}$. A 1-parameter family of functions $f_t \in \mathrm{Cyl}^2(\overline{\mathcal{A}})$ will be said to be a solution to the heat equation on $\overline{\mathcal{A}}$ if it satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} f_t = \Delta_{(l)} f_t\,. \tag{96}$$

Our main interest lies in defining a heat kernel for this equation. Recall first that the heat kernel ρ_t on G is the solution to the heat equation on G satisfying the initial condition $\rho_{t=0} = \delta(g, I_G)$, where I_G is the identity element of G and δ is the Dirac distribution. As is well known [23], for each t, ρ_t is a positive, smooth function on G. The next step is to use ρ_t to find a heat kernel $\rho_{\gamma,t}$ on $(\mathcal{A}_\gamma, \Delta_{\gamma,(l)})$. This $\rho_{\gamma,t}$ is a function on $\mathcal{A}_\gamma \times \mathcal{A}_\gamma$. Using a coordinate of $\mathcal{A}_\gamma$ induced by a group valued chart $\mathcal{A}_\gamma = G^E$, we can express $\rho_{\gamma,t}$ as

$$\rho_{\gamma,t}(A_\gamma, B_\gamma) = \rho_{s_1}(b_1^{-1}a_1) \ \dots \ \rho_{s_E}(b_E^{-1}a_E), \tag{97}$$

where $A_\gamma = (a_1, \dots, a_E)$, $B_\gamma = (b_1, \dots, b_E)$ and where s_i with $i = 1, \dots, E$ are positive numbers given by $s_i = l(e_i)t$. (Note that, since the heat kernel ρ_t on G is $\mathrm{Ad}(G)$-invariant, the above formula is independent of the choice of coordinates.) The family $(\rho_{\gamma,t})_{\gamma\in L}$ does not define a cylindrical function on $\overline{\mathcal{A}}$. However, we can consider the convolution of the heat kernel $\rho_{\gamma,t}$ with a function $f_\gamma \in \mathbb{C}^0(\mathcal{A}_\gamma)$. In the coordinate used above, this reads

$$(\rho_{t,\gamma} \star f_\gamma)(A_\gamma) = \int_{A_\gamma} \rho_{t,\gamma}(A_\gamma, B_\gamma) f(B_\gamma) \mu_{\mathrm{H}}(B_\gamma) \,, \tag{98}$$

where μ_{H} is the Haar measure on $\mathcal{A}_\gamma$. Now, the key point is that these convolutions do satisfy the consistency conditions

$$f_{\gamma_1} \sim f_{\gamma_2} \quad \Rightarrow \quad \rho_{t,\gamma_1} \star f_{\gamma_1} \sim \rho_{t,\gamma_1} \star f_{\gamma_1} \,. \tag{99}$$

Therefore, we *can* define a convolution map $\rho_t\star$ on $\overline{\mathcal{A}}$.

Theorem 6. *Let $\Delta_{(l)} = (\Delta_{\gamma,(l)})_{\gamma,\gamma'\in L}$ be the Laplace operator on $\overline{\mathcal{A}}$ given by the edge-metric l and let $(\rho_{\gamma,t})_{\gamma\in L}$ be the corresponding family of heat kernels, with $t \geq 0$. Then:*
(i) *there exists a linear map $\rho_t\star \ : \ \mathbb{C}^0(\overline{\mathcal{A}}) \to \mathbb{C}^0(\overline{\mathcal{A}})$ defined by*

$$\rho_t \star f = [\rho_{t,\gamma} \star f_\gamma]_\sim \ , \tag{100}$$

which is continuous with respect to the sup-norm;
(ii) *if $f \in \mathrm{Cyl}^2(\overline{\mathcal{A}})$ then*

$$f_t := \rho_t \star f \tag{101}$$

solves the heat equation with the initial value $f_{t=0} = f$;
(iii) *the map $\rho_t\star$ carries $\mathbb{C}^0(\overline{\mathcal{A}/\mathcal{G}})$ (the $\overline{\mathcal{G}}$-invariant elements of $\mathbb{C}^0(\overline{\mathcal{A}})$) into $\mathbb{C}^0(\overline{\mathcal{A}/\mathcal{G}})$.*

Proof. To establish (i), it is sufficient to prove the right-hand side of (100) is well defined. This would imply that $\rho_t^\star$ is well defined on $\mathbb{C}^0(\overline{\mathcal{A}})$. Continuity follows from the explicit formula for the convolution.

Let $f = [f_{\gamma_1}]_\sim = [f_{\gamma_2}]_\sim \in \mathrm{Cyl}^2(\overline{\mathcal{A}})$. Choose any $\gamma' \geq \gamma_1, \gamma_2$. Using the consistency conditions satisfied by the family $(\Delta_{\gamma,(l)})_{\gamma,\gamma'\in L}$ which are guaranteed by Theorem 5, and setting $i = 1, 2$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\, p^{\star}_{\gamma_i\gamma'}\,(\rho_{t,\gamma_i}\star f_{\gamma_i}) = p^{\star}_{\gamma_i\gamma'}\,(\Delta_{\gamma_i,(l)}\,\rho_{\gamma_i,t}\star f_{\gamma_i})$$
$$= \Delta_{\gamma',(l)}\, p^{\star}_{\gamma_i\gamma'}\,(\rho_{\gamma_i,t}\star f_{\gamma_i}) \qquad (102)$$

on $\mathcal{A}_{\gamma'}$. Thus, we conclude that the pull-backs of the two convolutions satisfy the heat equation on $\mathcal{A}_{\gamma'}$ with initial values $p^{\star}_{\gamma_1\gamma'}\, f_{\gamma_1}$ and $p^{\star}_{\gamma_2\gamma'} f_{\gamma_2}$ respectively. However, since $f_{\gamma_1}\sim f_{\gamma_2}$, the two initial values are equal. Hence, the two pulled-back convolutions on $\mathcal{A}_{\gamma'}$ are equal. Finally, since

$$p^{\star}_{\gamma_1\gamma'}\,(\rho_{t,\gamma_1}\star f_{\gamma_1}) \;=\; \rho_{t,\gamma'}\star p^{\star}_{\gamma_1\gamma'} f_{\gamma_1} \;=\; p^{\star}_{\gamma_2\gamma'}\,\rho_{t,\gamma_2}\star f_{\gamma_1}\,, \qquad (103)$$

we conclude that the right-hand side of (100) is well defined.

Part (ii) is now obvious and part (iii) follows from the $\overline{\mathcal{G}}$-symmetry of $\Delta^{(l)}$. □

We can now define the heat-kernel volume forms on $\overline{\mathcal{A}}$ associated to the Laplace operator Δ_l. These are the natural analogs of the Gaussian measures on Hilbert spaces, which can also be obtained as a projective limit of Gaussian volume forms on finite-dimensional subspaces. Now, on a finite-dimensional Hilbert space, the natural Gaussian functions $g(\vec{x})$ are given by the heat kernel, $g(\vec{x}) = \rho_t(o,\vec{x})$, and the Gaussian volume form is just the product of the translation invariant volume form by $g(\vec{x})$. The heat-kernel forms on $\mathcal{A}_\gamma$ will be obtained similarly by multiplying the Haar volume form by the heat kernel. Unlike a Hilbert space, however, $\mathcal{A}_\gamma$ does not have a preferred origin. Let us therefore first fix a point $\bar{A}_0 = (A_{0\gamma})_{\gamma\in L}\in\overline{\mathcal{A}}$. Then, on each $\mathcal{A}_\gamma$, we define a volume form ν_γ which, when evaluated at the point $A_\gamma\in\mathcal{A}_\gamma$, is given by

$$\nu_{t,\gamma}(A_\gamma) \;:=\; \rho_{t,\gamma}(A_{0\gamma}, A_\gamma)\mu^{\mathrm{H}}_\gamma(A_\gamma), \qquad (104)$$

where $\mu^{\mathrm{H}}_\gamma(A_\gamma)$ denotes the Haar form on $\mathcal{A}_\gamma$ at A_γ. We have the following theorem.

Theorem 7.

(i) *The family* $(\nu_{t,\gamma})_{\gamma\in L}$ *of volume forms defined by* (104) *is consistent;*

(ii) *the measure,* $\nu_t := (\nu_{t,\gamma})_{\gamma\in L}$, *defined on* $\overline{\mathcal{A}}$ *by the volume form is faithful iff for every nonzero edge* e *in* M, $l(e) > 0$.

The proof follows from Theorem 6, and from the nonvanishing of the heat kernel on G.

Thus, to define a heat-kernel volume form ν_t we have used the following input: an invariant metric tensor on G, an edge-metric l, a point $\bar{A}_0\in\overline{\mathcal{A}}$ and a "time" parameter $t > 0$. In terms of the heat kernel of $\Delta^{(l)}$, the integral of a function $\mathbb{C}^0(\overline{\mathcal{A}})\ni f = [f_\gamma]_\sim$ with respect to ν_t is given by the formula

$$\int_{\overline{\mathcal{A}}} f \;=\; (\rho_t\star f)(\bar{A}_0), \qquad (105)$$

which is completely analogous to the standard formula for integrals of functions on Hibert spaces with respect to the Gaussian measures.

Finally, let us consider the quotient $\overline{\mathcal{A}/\mathcal{G}} = \overline{\mathcal{A}}/\overline{\mathcal{G}}$. According to Theorem 6, the heat kernel ρ_t naturally projects to this quotient and hence we have a well-defined Gaussian measure (105) on $\overline{\mathcal{A}/\mathcal{G}}$. To have a $\overline{\mathcal{G}}$-invariant volume form on $\overline{\mathcal{A}}$ corresponding to this measure, we just average the Gaussian volume form on $\overline{\mathcal{A}}$, following the procedure of Section 5.1

$$\overline{\nu_t} = \int_{\overline{\mathcal{G}}} (R_{\bar{g}\star}\, \nu_t) \mathrm{d}\bar{g} , \tag{106}$$

where $\mathrm{d}\bar{g}$, as before, is the Haar form on $\overline{\mathcal{G}}$. The resulting volume form $\overline{\nu_t}$ shall be referred to as a Gaussian volume form on $\overline{\mathcal{A}/\mathcal{G}}$. Finally, on $\overline{\mathcal{A}/\mathcal{G}}$, there is a natural origin $[\bar{A}_0]$ whence the freedom is the choice of $\bar{A}_0$ can be eliminated. To see this, recall first that while constructing a heat-kernel measure on a group, one picks the identity element as the fiducial point. The obvious analog in the present case would be to choose $\bar{A}_0 \in \overline{\mathcal{A}}$ such that the parallel transport given by $\bar{A}_0$ (see (26)) along any closed loop in M is the identity. This does not single out $\bar{A}_0$ uniquely in $\overline{\mathcal{A}}$. However, all points $A_0 \in \overline{\mathcal{A}}$ with this property project down to a single point in the quotient $\overline{\mathcal{A}/\mathcal{G}} = \overline{\mathcal{A}}/\overline{\mathcal{G}}$.

The techniques introduced in this section can be extended in several directions. In the next subsection, we present one such extension, where heat-kernel measures are defined on the projective limit of the family associated with *noncompact* structure groups G. Another extension has been carried out in [8] where diffeomorphism invariant heat kernels are introduced using, in place of the induced Haar measure on $\overline{\mathcal{A}}$, the diffeomorphism invariant measures introduced by Baez [13,14].

6.3. Noncompact structure groups

We now wish to relax the assumption that the structure group be compact and let G be any connected Lie group. We will see that we *can* extend the construction of the induced Haar measure as well as the heat-kernel measures to this case, although there are now certain important subtleties.

To begin with, the construction of the space $\overline{\mathcal{A}/\mathcal{G}}$ given in [4] does not go through, since the Wilson loop functions are now unbounded and one cannot endow them with the structure of a $\mathbb{C}^\star$-algebra in a straightforward fashion. However, following Mourão and Marolf [21], we can still introduce the projective family $(\mathcal{A}_\gamma, \mathcal{G}_\gamma, p_{\gamma\gamma'})_{\gamma,\gamma' L}$ of analytic manifolds of Section 3. As in Section 3, the members $\mathcal{A}_\gamma$ of this family have the manifold structure of G^E (where E is the number of edges in γ) and are therefore no longer compact. Nonetheless, the projective limit is well defined and the notion of self-consistent families of measures $(\mu_\gamma)_{\gamma\in L}$ is still valid. Each such family enables us to integrate cylindrical functions. However, since the projective limit is no longer compact, the proof [6] that each self-consistent family defines a regular Borel measure μ on the limit does not go through. Thus, in general, the family only provides us with cylindrical measures on the projective limit. Nonetheless, the construction of these measures is nontrivial since the consistency conditions have to be solved and the result is both structurally interesting and practically useful.

Let us first construct some natural measures on $\overline{\mathcal{A}}$. In the previous constructions, we began with the Haar measure on G. However, since G is now noncompact, its Haar measure cannot be normalized and we need to use another measure as our starting point. Fix any probability measure $\mathrm{d}\mu_1$ on G which is $\mathrm{Ad}(G)$-invariant and has the following two properties:

$$\int_{G^2} f(gh)\,\mathrm{d}\mu_1(g)\,\mathrm{d}\mu_1(h) = \int_G f(g)\,\mathrm{d}\mu_1(g) \quad \text{and} \quad \mathrm{d}\mu_1(g) = \mathrm{d}\mu_1(g^{-1}) \tag{107}$$

for every $f \in \mathbb{C}^0_0(G)$ (the space of continuous functions on G with compact support). Then, the procedure introduced in [5] provides us with a family of measures $(\mu_\gamma)_{\gamma\in L}$, each μ_γ being the product measure on $\mathcal{A}_\gamma = G^E$. In spite of the fact that $\mathcal{A}_\gamma$ are noncompact, results of [5] required to ensure the self-consistency of the family are still applicable. We thus have an integration theory on $\overline{\mathcal{A}}$. The resulting cylindrical measure is again faithful and invariant under the induced action of $\mathrm{Diff}(M)$.

Another possibility is to use the Baez construction [13] which leads to a diffeomorphism invariant integration on $\overline{\mathcal{A}/\mathcal{G}}$ from almost any measure on G. The resulting cylindrical measures, however, fail to be faithful.

A third possibility is to repeat the procedure of gluing measures on $\mathcal{A}_\gamma$ starting, however, from an appropriately generalized heat-kernel measure on G. Given a measure μ on G, let us define an integral kernel, ρ via

$$(\rho_t \star f)(g) := \int_G f(gh^{-1})\,\mathrm{d}\mu(h)\,. \tag{108}$$

A 1-parameter family of measures μ_t on G will be said to constitute a *generalized heat-kernel measure* if the resulting integral kernels ρ_t satisfy the following conditions:

$$\rho_r \star \rho_s \star f = \rho_{r+s} \star f,\quad \rho_t \star R_{g\star} f = R_{g\star}\rho_t \star f,\quad \rho_t \star I_\star f = I_\star \rho_t \star f\,, \tag{109}$$

where R and I denote, respectively, the right action of G in G and the inversion map $g \mapsto g^{-1}$, and where $f \in \mathbb{C}^0_0(G)$.

Given an edge-metric l on M, we can now repeat the construction of the previous subsection and obtain a self-consistent family of measures $(\mu_\gamma)_{\gamma\in L}$. Thus, for a graph γ, define the following "generalized heat evolution"

$$(\rho_{t,\gamma} \star f_\gamma)(A_\gamma) = \int_{\mathcal{A}_\gamma} f_\gamma(a_1 b_1^{-1}, \dots, a_E b_E^{-1})\,\mathrm{d}\mu_{s_1}(b_1)\cdots\mathrm{d}\mu_{s_E}(b_E), \tag{110}$$

where $s_i = l(e_i)t$ and where we have set $A_\gamma = (a_1, \dots, a_E)$, $B_\gamma = (b_1, \dots, b_E)$ using an identification $\mathcal{A}_\gamma = G^E$. It is then straightforward to establish the following result.

Proposition 9. *The family of heat evolutions* (110) *is self-consistent, i.e., if f is a cylindrical function, $f = [f_{\gamma_1}]_\sim = [f_{\gamma_2}]_\sim$, then*

$$[\rho_{t,\gamma_1} \star f_{\gamma_1}]_\sim = [\rho_{t,\gamma_2} \star f_{\gamma_2}]_\sim. \tag{111}$$

Using the resulting heat evolution for $\overline{\mathcal{A}}$ we can just define the corresponding heat kernel integral to be

$$\int_{\overline{\mathcal{A}}} = \rho_t \star f(\mathcal{A}_0) \tag{112}$$

as before. If the group G is compact, this formula defines a regular Borel measure on $\overline{\mathcal{A}}$. In the general case, we only have a cylindrical measure. A case of special interest is when G is the complexification of a compact, connected Lie group. It is discussed in detail in [8].

7. Natural geometry on $\overline{\mathcal{A}/\mathcal{G}}$

Let us begin by spelling out what we mean by "natural structures" on $\overline{\mathcal{A}}$ and on $\overline{\mathcal{A}/\mathcal{G}}$.

A natural structure on $\overline{\mathcal{A}}$ will be taken to be a structure which is invariant under the induced action of the automorphism group of the underlying bundle $B(M, G)$. (Note that *all* automorphisms are now included; not just the vertical ones.) In the context of the quotient $\overline{\mathcal{A}/\mathcal{G}}$ a natural structure will have to be invariant under the induced action of the group Diff(M) of diffeomorphisms on M.

The action of Diff(M) on $\overline{\mathcal{A}/\mathcal{G}}$ may be viewed as follows. Let $\Phi \in$ Diff(M). Given a graph $\gamma = \{e_1, \ldots, e_E\}$ choose any orientation of the edges and consider the graph $\Phi(\gamma) = \{\Phi(e_1), \ldots, \Phi(e_E)\}$ with the corresponding orientation. Pick any trivialization of $B(M, G)$ over the vertices of γ and $\Phi(\gamma)$ and define

$$\Phi_\gamma : \mathcal{A}_\gamma/\mathcal{G}_\gamma \rightarrow \mathcal{A}_{\Phi(\gamma)}/\mathcal{G}_{\Phi(\gamma)}, \qquad \Phi_\gamma := \Lambda^{-1}_{\Phi(\gamma)} \circ \Lambda_\gamma, \tag{113}$$

where Λ_γ is the group valued chart defined in (17). It is easy to see that the family of maps Φ_γ defines uniquely a map

$$\overline{\Phi} : \overline{\mathcal{A}/\mathcal{G}} \rightarrow \overline{\mathcal{A}/\mathcal{G}}. \tag{114}$$

This is the action of Diff(M) on $\overline{\mathcal{A}/\mathcal{G}}$.

In this section we will introduce a natural contravariant, second rank, symmetric tensor field k_0 on $\overline{\mathcal{A}/\mathcal{G}}$ and, using it, define a natural second-order differential operator. In the conventional approaches, by contrast, the introduction of such Laplace-type operators on (completions of) $\mathcal{A}/\mathcal{G}$ always involve, to our knowledge, the use of a background metric on M (see, e.g. [12]).

7.1. Cometric tensors

Let us suppose that we are given, at each point A_γ of $\mathcal{A}_\gamma$, a real, bilinear, symmetric, contravariant tensor k_γ:

$$k_\gamma : \bigwedge{}_{A_\gamma} \times \bigwedge{}_{A_\gamma} \rightarrow \mathbb{R}, \tag{115}$$

for all graphs γ, where $\bigwedge_{A_\gamma}$ denotes the cotangent space of $\mathcal{A}_\gamma$ at A_γ. Such a family, $(k_\gamma)_{\gamma\in L}$, will be called a *cometric tensor* on $\overline{\mathcal{A}}$ if it satisfies the following consistency condition:

$$k_{\gamma'}(p^*_{\gamma\gamma'}\phi_\gamma \wedge p^*_{\gamma\gamma'}\nu_\gamma) = k_\gamma(\phi_\gamma \wedge \nu_\gamma), \tag{116}$$

for every $\phi_\gamma, \nu_\gamma \in \bigwedge_{A_\gamma}(\mathcal{A}_\gamma)$, whenever $\gamma' \geq \gamma$.

Example 10. Recall from the remark at the end of Section 6.1 that, given a path metric l, and an edge e, one can introduce on $\mathcal{A}_e$ a contravariant metric $l(e)k_e$, where k_e is the contravariant metric induced on $\mathcal{A}_e$ by a fixed Killing form on G. Then, given a graph γ, $\mathcal{A}_\gamma = \times_{e\in\gamma}\mathcal{A}_e$ is equipped with the product contravariant metric

$$k^{(l)}_\gamma := \sum_{e\in\gamma} l(e)k_e.$$

It is not hard to check that $(k^{(l)}_\gamma)_{\gamma\in L}$ is a cometric $k^{(l)}$ on $\overline{\mathcal{A}}$. Notice, that if the edge metric vanishes for some edges e_0, the cometric is degenerate but continues to be well defined.

Given a cometric $k = (k_\gamma)_{\gamma\in L}$ and a differential 1-form $\phi = [\phi_\gamma]_\sim \in \Omega^1(\overline{\mathcal{A}})$, we define for each ϕ_γ the vector field $k_\gamma(\phi_\gamma)$ on $\mathcal{A}_\gamma$ determined by the condition that, for any 1-form ν on $\mathcal{A}_\gamma$,

$$\nu(k_\gamma(\phi_\gamma)) = k_\gamma(\nu, \phi_\gamma)$$

at every $A_\gamma \in \mathcal{A}_\gamma$. It is easy to see that the family of vector fields $(k_{\gamma'}(\phi_{\gamma'}))_{\gamma'\geq\gamma} =: k(\phi)$ defines a vector field on $\overline{\mathcal{A}}$ in the sense of Section 4. Let us suppose that we are given a cometric tensor k and a 1-form ϕ on $\overline{\mathcal{A}}$ both of which are $\overline{\mathcal{G}}$-invariant. Then so is the vector field $k(\phi)$. Hence, we can apply the results of Section 5 to obtain the following proposition.

Proposition 10. *Suppose k is a $\overline{\mathcal{G}}$-invariant cometric tensor on $\overline{\mathcal{A}}$ and $\phi \in \Omega^1(\overline{\mathcal{A}/\mathcal{G}})$. Then the vector field $k(\phi)$ is compatible with the natural Haar volume form μ_0 on $\overline{\mathcal{A}}$.*

Hence, assuming the necessary differentiability, $k(\phi)$ has a well-defined divergence with respect to the Haar volume form μ_0 on $\overline{\mathcal{A}}$, $\mathrm{div}_{\mu_0} k(\phi) = [\mathrm{div} k_\gamma(\phi_\gamma)]_\sim$. Consequently, with every $\overline{\mathcal{G}}$-invariant cometric on $\overline{\mathcal{A}}$ we may associate a second-order differential operator

$$\Delta^{(k)} : \mathrm{Cyl}^2(\overline{\mathcal{A}/\mathcal{G}}) \mapsto \mathrm{Cyl}^0(\overline{\mathcal{A}/\mathcal{G}}); \quad \Delta^{(k)} f = \mathrm{div}_{\mu_0}[k(\mathrm{d}f)] \tag{117}$$

with a well-defined action on the space of $\overline{\mathcal{G}}$-invariant cylindrical functions. If k_γ is nondegenerate, $\Delta^{(k)}$ is the Laplacian it defines. (In particular, the operator defined by the cometric $k^{(l)}$ via (117) is precisely the Laplacian constructed from the edge-metric l in Section 6.) Hence, for simplicity of notation, we will refer to $\Delta^{(k)}$ as the *Laplace operator corresponding to k* even in the degenerate case.

7.2. A natural cometric tensor

We will now show that $\overline{\mathcal{A}/\mathcal{G}}$ admits a *natural* (nontrivial) cometric. Let us begin by fixing an invariant contravariant metric on the gauge group G (i.e., an invariant scalar product in the cotangent space at each point of G). This is the only "input" that is needed. The idea is to use this k to assign a scalar product k_v to each vertex v of a graph γ and set $k_\gamma = \sum_v k_v$.

Given a vertex v, choose an orientation of γ such that all the edges at v are outgoing. Use a group valued chart Λ_γ (see (17)) to define for each edge e in γ a map

$$G \rightarrow \mathcal{A}_e \rightarrow \times_{e'\in\gamma}\mathcal{A}_{e'}. \tag{118}$$

Through the corresponding pull-back map every $\phi \in \bigwedge_{A_\gamma}$ defines a 1-form ϕ_e on G for every edge e. Let k_e and $k_{ee'}$ be two bilinear forms in $\bigwedge_{A_\gamma}$ defined by

$$k_e(\phi, \nu) = k(\phi_e, \nu_e), \qquad k_{ee'}(\phi, \nu) = k(\phi_e, \nu_{e'}) . \tag{119}$$

(Note that k_e is as in the example in the last subsection.) Both, k_e and $k_{ee'}$ are insensitive to a change of a group valued chart since the field k on G is left and right invariant. Now, for k_v we set

$$k_v := \frac{1}{2}\sum_e k_e + \frac{1}{2}\sum_{ee'} k_{ee'}, \tag{120}$$

where the first sum is carried over all the edges passing through v and, in the second, e, e' range through all the pairs of edges which form an analytic arc at v (i.e, such that $e' \circ e^{-1}$ is analytic at v). Finally, we define k_γ simply by summing over all the vertices v of γ:

$$k_\gamma := \sum_v k_v . \tag{121}$$

We then have the following proposition.

Proposition 11. *The family $k_0 = (k_\gamma)_{\gamma\in L}$ is a natural cometric tensor on $\overline{\mathcal{A}}$; k_0 is $\overline{\mathcal{G}}$-invariant, and defines a natural cometric on $\overline{\mathcal{A}/\mathcal{G}}$.*

The proof consists just of checking the consistency conditions and proceeds as the previous proofs of this property. Note that consistency holds only because we have added the terms $k_{ee'}$, assigned to each pair of edges which constitute a single edge of a smaller graph. (Unfortunately, however, this is also the source of the potential degeneracy of k_0.) The diffeomorphism invariance of k_0 is obvious.

7.3. A natural Laplacian

We can now use the natural cometric to define a natural Laplacian. That is, using Propositions 10 and 11 and Eq. (117), we can assign to k_0 the operator $(\Delta^0 \equiv \Delta^{(k_0)}, \mathrm{Cyl}^2(\overline{\mathcal{A}/\mathcal{G}}))$. In this subsection, we will write out its detailed expression and discuss some of its properties.

Let us fix a group valued chart on $\mathcal{A}_\gamma$. The operator Δ^0_γ representing Δ^0 in $\mathbb{C}^2(\mathcal{A}_\gamma/\mathcal{G}_\gamma)$ is given by the sum

$$\Delta^0_\gamma = \sum_v \Delta_v, \tag{122}$$

where each Δ_v will involve products of derivatives on the copies $\mathcal{A}_e$ corresponding to edges incident at v. To calculate Δ_v, we will orient γ such that the edges at v are all outgoing. Let R_i, $i = 1, \ldots, n$ be a basis of left-invariant vector fields on G, θ^i the dual basis and let $k^{ij} = k(\theta^i, \theta^j)$ denote the components of k in this basis. Let e be an edge with vertex v. Denote by R_{ei} the vector field on $\mathcal{A}_e$ corresponding to R_i via the group valued chart. Next, identify R_{ei} with the vector field $(0, \ldots, 0, R_{ei}, 0, \ldots, 0)$ on $\mathcal{A}_\gamma = \times_{e' \in \gamma} \mathcal{A}_{e'}$. Then, the expression of Δ_v reads

$$\Delta_v = \frac{1}{2} \sum_e k^{ij} R_{ei} R_{ej} + \frac{1}{2} \sum_{ee'} k^{ij} R_{ei} R_{e'j}\ , \tag{123}$$

where the sums are as in the expression (120) of k_v.

The consistency of the family of operators $(\Delta^0_\gamma, \mathrm{Cyl}^2(\mathcal{A}_\gamma/\mathcal{G}_\gamma))$ defined by (122) and (123) is assured by Theorem 3. However, one can also verify the consistency directly from these two equations. Indeed, if a vertex v belongs to only two edges, say e and e', which form an analytic arc (and are oriented to be outgoing), then

$$R_{ei} f_\gamma = -R_{e'i} f_\gamma \tag{124}$$

so that $\Delta_v = 0$ at this vertex v. This ensures

$$p_{\gamma\gamma'\star} \Delta^0_{\gamma'} = \Delta^0_\gamma, \tag{125}$$

when γ' is obtained from γ by splitting an edge. On the other hand, if γ' is constructed by adding an extra edge, say e'', to γ then the projection kills every term containing $R_{e''i}$. The remaining terms coincide with those of Δ_γ. Furthermore, in this argument, we need not restrict ourselves to $\mathcal{G}_\gamma$-invariant functions. This shows that the vector field $k_0(\mathrm{d}f)$ is compatible with the natural volume form μ_0 on $\overline{\mathcal{A}}$ for *every* $f \in \mathrm{Cyl}^2(\overline{\mathcal{A}})$. Hence, the operator

$$\Delta^0 : \mathrm{Cyl}^2(\overline{\mathcal{A}}) \mapsto \mathrm{Cyl}(\overline{\mathcal{A}}); \qquad \Delta^0 f = \mathrm{div}_{\mu_0}[k_0(\mathrm{d}f)] \tag{126}$$

is also well defined.

As in Section 6, the Laplace operator Δ^0 can be used to define a semigroup of transformations $\rho_{t\star} : \mathrm{Cyl}(\overline{\mathcal{A}}) \to \mathrm{Cyl}(\overline{\mathcal{A}})$ such that $f_t := \rho_{t\star} f$ solves the corresponding heat equation. In this case, $\rho_{t\star}$ is given by the family $(\rho_\gamma)_{\gamma\in L}$ of certain generalized heat kernels. The family and the transformations $\rho_{t\star}$ coincide with those introduced in [8]. In fact the constructions of this subsection were motivated directly by the results of [8].

We will conclude with two remarks.

(1) If a vector field X on $\overline{\mathcal{A}}$ is compatible with a volume form μ, then so is $Y = hX$ for any $h \in \mathrm{Cyl}^1(\overline{\mathcal{A}})$ (and $\mathrm{div}(Y) = h\,\mathrm{div}(X) + X(h)$). From this and from the existence of the operator (126), we have the following proposition.

Proposition 12. *For every (differentiable) 1-form $\phi \in \Omega^1(\overline{\mathcal{A}})$, the vector field $k_0(\phi)$ is compatible with the Haar volume form μ_0.*

The map

$$d^\star : \Omega^1(\overline{\mathcal{A}}) \mapsto \Omega^0(\overline{\mathcal{A}}), \qquad d^\star\phi = \mathrm{div}_{\mu_0}[k_0(\phi)]$$

can be thought of as a coderivative defined on $\overline{\mathcal{A}}$ by the geometry (k_0, μ_0). It is possible to extend $d^\star$ to act on $\Omega^n(\overline{\mathcal{A}})$.

(2) Consider the special case when the structure group G is $SU(2)$ and M is an oriented three-dimensional manifold. There exists on $\overline{\mathcal{A}}$ a natural *third-order* differential operator, $(q,\ \mathrm{Cyl}^3(\overline{\mathcal{A}}))$ defined by a consistent family of operators $(q_\gamma, \mathbb{C}^3(\mathcal{A}_\gamma))_{\gamma\in L}$ on $\mathbb{C}^3(\mathcal{A}_\gamma)$. To obtain q_γ, we begin as before, by defining operators q_v associated with the vertices of γ. Given a vertex v of a graph γ and a triplet of edges (e, e', e'') coincident on this vertex, let $\epsilon(e, e', e'')$ be 0 whenever their tangents at v are linearly dependent, and ± 1 otherwise, depending on the orientation of the ordered triplet. To the vertex v, let us assign an operator acting on $\mathbb{C}^3(\mathcal{A}_\gamma)$ as

$$q_v = \frac{1}{(3!)^2}\ \mu_{\mathrm{H}}(R_i, R_j, R_k) \sum_{e,e',e''} \epsilon(e, e', e'')\ R_{ei} R_{e'j} R_{e''k}, \tag{127}$$

where we use the same orientation, notation and group valued charts as in (123), and where the summation ranges over all the ordered triplets (e, e', e'') of edges incident at v. In terms of these operators, we can now define q_γ. Set

$$q_\gamma := \sum_v |q_v|^{1/2} \tag{128}$$

where v runs through all the vertices of γ (at which three or more vertices have to be incident to contribute). We then have the following proposition.

Proposition 13. *The family $(q_\gamma, \mathbb{C}^3(\mathcal{A}_\gamma))_{\gamma\in L}$ of operators is self-consistent. The operator $(q, \mathrm{Cyl}^3(\overline{\mathcal{A}}))$ is natural, $\overline{\mathcal{G}}$-invariant and defines a natural operator on $\mathrm{Cyl}^3(\overline{\mathcal{A}/\mathcal{G}})$.*

This operator is closely related to the total volume (of M) operator of Riemannian quantum gravity [22,9]. The detailed derivation of this operator as well as the local area and volume operators and the analytic formulae for the Hamiltonian operators will be discussed in a forthcoming paper [10,20].

8. Discussion

In more conventional approaches to gauge theories, one begins with $\mathcal{A}/\mathcal{G}$, introduces suitable Sobolev norms and completes it to obtain a Hilbert–Riemann manifold (see, e.g., [12]). However, since the Sobolev norms use a metric on the underlying manifold M the resulting Hilbert–Riemann geometry fails to be invariant with respect to induced action of Diff(M). Hence, to deal with diffeomorphism invariant theories such as general relativity, a new approach is needed, an approach in which the basic constructions are not tied to a background structure on M. Such a framework was introduced in [4,5] and extended in [13,6] to a theory of diffeomorphism invariant measures. Here, the basic assumption is that the algebra of Wilson loop variables should be faithfully represented in the quantum theory and it leads one directly to the completion $\overline{\mathcal{A}/\mathcal{G}}$. Neither the construction of $\overline{\mathcal{A}/\mathcal{G}}$ nor the introduction of a large family of measures on it requires a metric on M. This is a key strength of the approach. The physical concepts it leads to are also significantly different from the perturbative treatments of the more conventional approaches: loopy excitations and flux lines are brought to forefront rather than wave-like excitations and notion of particles. The framework *has been* successful in dealing with Yang–Mills theory in two space–time dimensions [7] without a significant new input. This success may, however, be primarily due to the fact that in two dimensions, the Yang–Mills theory is invariant under all volume preserving diffeomorphisms. In higher dimensions, it remains to be seen whether the Yang–Mills theory admits interesting phases in which the algebra of the Wilson loop operators is faithfully represented. If it does, they would be captured on $\overline{\mathcal{A}/\mathcal{G}}$, e.g., through Laplacians and measures which depend on an edge-metric $l(e)$, which itself would be constructed from a metric on M. We expect, however, that the key applications of the framework would be to diffeomorphism invariant theories.

A central mathematical problem for such theories is that of developing differential geometry on the quantum configuration space, again without reference to a background structure on M. This task was undertaken in the last four sections. In particular, we have shown that although $\overline{\mathcal{A}/\mathcal{G}}$ initially arises only as a compact Hausdorff topological space, because it can be recovered as a projective limit of a family of *analytic manifolds*, using algebraic methods, one can induce on it a rich geometric structure. There are strong indications that this structure will play a key role in nonperturbative quantum gravity. Specifically, it appears that results of this paper can be used to give a rigorous meaning to various operators that have played an important role in various heuristic treatments [2,22].

We will conclude by indicating how the results of this paper fit in the general picture provided by the available literature on the structure of $\overline{\mathcal{A}/\mathcal{G}}$.

As mentioned in Section 1, $\overline{\mathcal{A}/\mathcal{G}}$ first arose as the Gel'fand spectrum of an Abelian $\mathbb{C}^\star$-algebra $\overline{\mathcal{HA}}$ constructed from the so-called Wilson loop functions on the space $\mathcal{A}/\mathcal{G}$ of smooth connections modulo gauge transformations [4]. A complete and useful characterization of this space [5] can be summarized as follows. Fix a base point x_0 on the underlying manifold M and consider piecewise analytic, closed loops beginning and ending at x_0. Consider two loops as equivalent if the holonomies of any smooth connection around them, evaluated at x_0, are equal. Call each equivalence class a *hoop* (a holonomic-loop). The space $\mathcal{HG}$ of hoops has the structure of a group, which is called the *hoop group*. (It turns out that in the piecewise analytic case, $\mathcal{HG}$ is largely independent of the structure group G. More precisely, there are only two hoop groups; one for the case when G is Abelian and the other when it is non-Abelian.) The characterization of $\overline{\mathcal{A}/\mathcal{G}}$ can now be stated quite simply, in purely algebraic terms: $\overline{\mathcal{A}/\mathcal{G}}$ is the space of all homomorphisms from $\mathcal{HG}$ to G. Using subgroups of $\mathcal{HG}$ which are generated by a finite number of independent hoops, one can then introduce the notion of cylindrical functions on $\overline{\mathcal{A}/\mathcal{G}}$. (The $\mathbb{C}^\star$-algebra of all continuous cylindrical functions turns out to be isomorphic with the holonomy $\mathbb{C}^\star$-algebra with which we began.) Using these functions, one can define cylindrical measures. The Haar measure on G provides a natural cylindrical measure, which can be then shown to be a regular Borel measure on $\overline{\mathcal{A}/\mathcal{G}}$ [5].

Marolf and Mourão [21] obtained another characterization of $\overline{\mathcal{A}/\mathcal{G}}$: using various techniques employed in [5], they introduced a projective family of compact Hausdorff spaces and showed that its projective limit is naturally isomorphic to $\overline{\mathcal{A}/\mathcal{G}}$. This result influenced the further developments significantly. Indeed, as we saw in this paper, it is a projective limit picture of $\overline{\mathcal{A}/\mathcal{G}}$ that naturally leads to differential geometry. The label set of the family they used is, however, different from the one we used in this paper: it is the set of all finitely generated subgroups of the hoop group. At a sufficiently general level, the two families are equivalent, the subgroups of the hoop group being recovered in the second picture as the fundamental groups of graphs. For a discussion of measures and integration theory, the family labelled by subgroups of the hoop group is just as convenient as the one we used. Indeed, it was employed successfully to investigate the support of the measure μ_0 in [21] and to considerably simplify the proofs of the two characterization of $\overline{\mathcal{A}/\mathcal{G}}$, discussed above, in [6]. For introducing differential geometry, however, this projective family appears to be less convenient.

The shift from hoops to graphs was suggested and explored by Baez [13,14]. Independently, the graphs were introduced to analyze the so-called "strip-operators" (which serve as gauge-invariant momentum operators) as a consistent family of vector fields in [19]. Baez also pointed out that, even while dealing with gauge-invariant structures, it is technically more convenient to work "upstairs", in the full space of connections, rather than in the space of gauge equivalent ones. Both these shifts of emphasis led to key simplifications in the various constructions of this paper.

Baez's main motivation, however, came from integration theory. His main result was two-fold: he discovered powerful theorems that simplify the task of obtaining measures and, using them, obtained a large class of diffeomorphism invariant measures on $\overline{\mathcal{A}/\mathcal{G}}$. In his discussion, it turned out to be convenient to deemphasize $\overline{\mathcal{A}/\mathcal{G}}$ itself and focus instead on the space $\mathcal{A}$ of smooth connections. In particular, he regarded measures on $\overline{\mathcal{A}}$ primarily as defining "generalized measures" on $\mathcal{A}$. At first this change of focus appears to simplify the matters considerably since one seems to be dealing only with the familiar space of *smooth* connections. One may thus be tempted to ignore $\overline{\mathcal{A}/\mathcal{G}}$ altogether!

The impression is, however, misleading for a number of reasons. First, as Marolf and Mourão [21] have shown, the support of the natural measure μ_0 is concentrated precisely on $\overline{\mathcal{A}}$–$\mathcal{A}$; the space $\mathcal{A}$ is contained in a set of zero μ_0-measure. The situation is likely to be the same for other interesting measures on $\overline{\mathcal{A}/\mathcal{G}}$. Thus, the extension from $\mathcal{A}$ to $\overline{\mathcal{A}}$ is not an irrelevant technicality. Second, without recourse to $\overline{\mathcal{A}}$, it is difficult to have a control on just how general the class of "generalized measures" on $\mathcal{A}$ is. Is it perhaps "too general" to be relevant to physics? A degree of control comes precisely from the fact that this class corresponds to regular Borel measures on $\overline{\mathcal{A}}$. One can thus rephrase the question of relevance of these measures in more manageable terms: Is $\overline{\mathcal{A}}$ "too large" to be physically useful and mathematically interesting? We saw in this paper that, with this rephrasing, the question can be analyzed quite effectively. The answer turned out to be in the negative; although $\overline{\mathcal{A}}$ is very big, it is small enough to admit a rich geometry. Finally, all our geometrical structures naturally reside on the projective limit $\overline{\mathcal{A}}$ of our family of compact analytic manifolds. It would have been difficult and awkward to analyze them directly as generalized structures on $\mathcal{A}$. Thus, there is no easy way out of making the completions $\overline{\mathcal{A}}$ and $\overline{\mathcal{A}/\mathcal{G}}$.

To summarize, for diffeomorphism invariant theories, there is no easy substitute for the extended spaces $\overline{\mathcal{A}}$ and $\overline{\mathcal{A}/\mathcal{G}}$; one has to learn to deal directly with *quantum* configuration spaces. Fortunately, this task is made manageable because there are three characterizations of $\overline{\mathcal{A}/\mathcal{G}}$: One as a Gel'fand spectrum of an Abelian $\mathbb{C}^\star$-algebra and two as projective limits. The three ways of constructing $\overline{\mathcal{A}/\mathcal{G}}$ are complementary, and together they show that $\overline{\mathcal{A}/\mathcal{G}}$ has a surprisingly rich structure. In particular, differential geometry developed in this paper makes it feasible to use $\overline{\mathcal{A}/\mathcal{G}}$ to analyze concrete mathematical problems of diffeomorphism invariant theories.

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